Naval Research Laboratory
Washington, DC 20375-5000

NRL Report 9115

Adaptive Canceller Limitations Caused by I,Q Mismatch Errors

KARL GERLACH

Target Characteristics Branch
Radar Division

September 5, 1989

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1a. REPORT SECURITY CLASSIFICATION: UNCLASSIFIED	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): NRL Report 9115	5. MONITORING ORGANIZATION REPORT NUMBER(S)
6a. NAME OF PERFORMING ORGANIZATION: Naval Research Laboratory; 6b. OFFICE SYMBOL (If applicable): 5340.1G	7a. NAME OF MONITORING ORGANIZATION
6c. ADDRESS (City, State, and ZIP Code): Washington, DC 20375-5000	7b. ADDRESS (City, State, and ZIP Code)
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: Office of Naval Technology; 8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
8c. ADDRESS (City, State, and ZIP Code): Arlington, VA 22217-5000	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. (See page ii); PROJECT NO.; TASK NO.; WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
Adaptive Canceller Limitations Caused by I,Q Mismatch Errors

12. PERSONAL AUTHOR(S)
Gerlach, Karl

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Interim	FROM ______ TO ______	1989 September 5	39

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Adaptive filter; Adaptive cancellation; Radar; ECCM

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The effects of I and Q phase, amplitude, and low pass filter (LPF) errors on adaptive cancellers are investigated. I,Q errors occur because of errors in the synthesis process of the mixers and LPFs designed to be identical for each input channel. These I,Q errors among the channels result in cancellation degradation. Tapped delay line transversal filters have been proposed as a way to compensate for these errors and thus improve cancellation performance. However, it is shown that if there is any LPF mismatch, then transversal filtering has a small effect on improving canceller performance. The method of individual I,Q adaptive transversal filter weighting is suggested as a means of eliminating the phase and amplitude errors and making the canceller performance responsive to transversal filter compensation. In addition, the cancellation performance of cascaded mismatched IF and I,Q filters is briefly considered.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: UNCLASSIFIED
22a. NAME OF RESPONSIBLE INDIVIDUAL: Karl Gerlach	22b. TELEPHONE (Include Area Code): (202) 767-3599; 22c. OFFICE SYMBOL: 5340.1G

DD Form 1473, JUN 86 — Previous editions are obsolete. — SECURITY CLASSIFICATION OF THIS PAGE

PROGRAM ELEMENT	PROJECT NO.	TASK NO.	WORK UNIT NO.ACCESSION NO.
6271N		RS12-131-001	DN480-54
		XF12-141-100	DN380-10

CONTENTS

ADAPTIVE CANCELLER LIMITATIONS CAUSED BY I,Q MISMATCH ERRORS

I. INTRODUCTION

An adaptive canceller combines auxiliary channels of data with a main channel of data in such a way so as to minimize the main channel output noise power residue. Hence, an effective way of eliminating unwanted data (or noise) from a main channel (the information channel) is by inputting correlated data from auxiliary channels. Mismatch errors of any kind between channels of an adaptive canceller can cause a reduction in the achievable cancellation ratio. These mismatch errors can include small time-delay differences, in-phase (I) and quadature-phase (Q) imbalances, sampling errors, and filter frequency mismatch errors among the various channels. For a radar or communications digital canceller, many of these errors occur because of the radio frequency (RF)-to-intermediate frequency (IF)-to-baseband-to-sample and hold (S+H)-to-analog-to-digital (A/D) chain that is present in each channel as illustrated in Fig. 1(a). If any link of this chain is not identical among the channels, mismatch errors cause the canceller performance to degrade.

This report is an extension of Ref. 1 in which the effects of IF filter mismatch errors on adaptive cancellers were investigated. In this report, the effects of I,Q phase and amplitude quadrature errors and low pass filter (LPF) mismatches on adaptive cancellation performance are examined. We also briefly discuss the effects of cascading mismatched IF and I,Q filters. Additional research in this area can be found in Refs. 2 and 3.

To compensate for frequency mismatch errors, adaptive digital transversal filters are often inserted into the auxiliary channels. Figure 1(b) illustrates a two-channel compensated adaptive canceller. Here, we have two signals $y_M(t)$ and $y_A(t)$ inputted to the main and auxiliary channels, respectively. These signals may be at RF or IF. Each signal is quadrature detected into I and Q components through a double mixer operation. Thereafter, each I and Q component is passed through a low pass filter to eliminate upper band components while retaining the baseband information. The four resulting channels are sampled every T seconds and converted into digital form.

Errors occur in the mixer operation in the form of amplitude and phase perturbations, i.e., the amplitude and phases of each mixer may not be identical. In addition, mismatch errors occur in the synthesis of the LPFs. Normally, these are designed to be identical but because of inaccuracies in the synthesis process, the LPFs are rarely identical.

To compensate for this mismatch, an adaptive transversal filter (or a tapped delay line) is often inserted into the auxiliary channel, and weights w_n, $n = 1, 2, \ldots, N$ on these taps are adjusted so that the output noise power residue of $v(t)$ (see Fig. 1(b)) is minimized. Note that the tap time delay T is normally less than the Nyquist sampling interval $1/B$, where B is the input signal's bandwidth (includes $\pm$ frequencies). In addition, the main channel is delayed such that the auxiliary samples are time-centered.

Manuscript approved December 8, 1988.

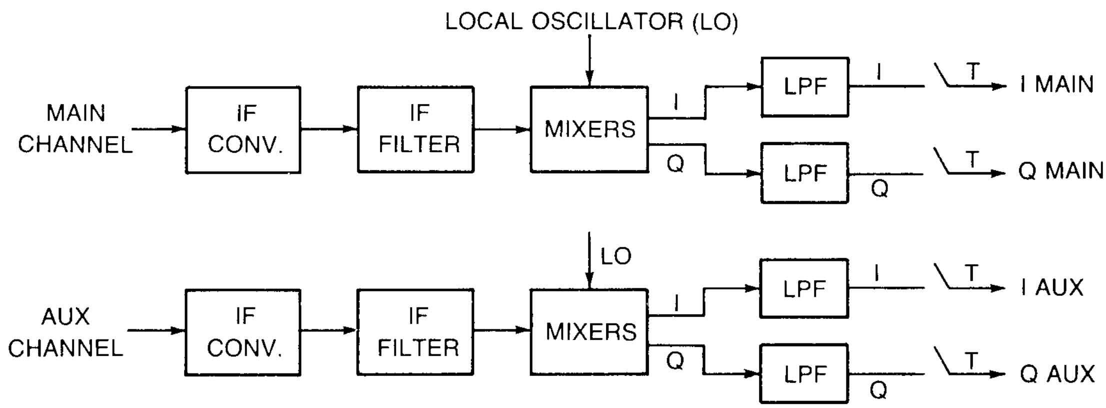

Fig. 1(a) — RF-to-IF-to-baseband-to-digital conversion chain for the main and auxiliary channels

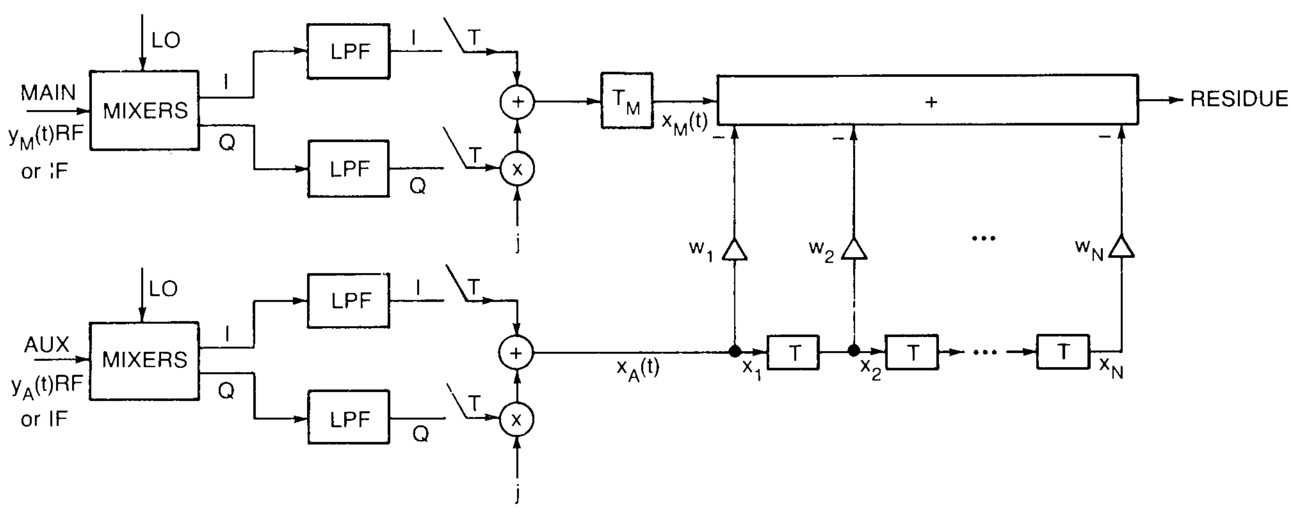

Fig. 1(b) — I,Q conversion and adaptive transversal filter compensation

If we define $\mathbf{w} = (w_1, w_2, \ldots, w_N)^{\mathrm{T}}$ to be the optimal complex valued weighting vector, where T denotes the transpose operation, then it can be shown [2] that $\mathbf{w}$ is the solution of the vector equation:

$$\mathbf{R}\,\mathbf{w} = \mathbf{r}, \tag{1.1}$$

where $\mathbf{R}$ is the covariance matrix of the time-delayed taps in the auxiliary channel and $\mathbf{r}$ is the cross covariance vector between the auxiliary taps and the time-centered main channel. More formally

$$\mathbf{R} = E\{\mathbf{x}^* \, \mathbf{x}^{\mathrm{T}}\} \tag{1.2}$$

and

$$\mathbf{r} = E\{\mathbf{x}^* \, x_M\}, \tag{1.3}$$

where $E\{\cdot\}$ denotes the expected value, * denotes the complex conjugate, and $\mathbf{x} = (x_1, x_2, \ldots, x_N)^{\mathrm{T}}$ is the vector of time-delayed signals in the auxiliary channel.

To completely understand the effects of the frequency mismatch errors, the statistical characteristics of the input signals in the main and auxiliary channels must be known. However, this may not be possible in many instances. We have chosen to characterize and investigate the effects of the frequency mismatch errors on cancellation when the adaptive canceller is in the self-cancellation mode. Here, we tie the main and auxiliary inputs together and input a wideband signal. We then calculate first order error expressions for the output cancellation power residue. Thus the self-canceller mode yields best case (or an upper bound on) cancellation performance.

This report is laid out as follows: The phase and amplitude quadrature error model and the LPF error model are presented in Section II. A "separation principle" that allows us to separate the phase and amplitude errors from the LPF errors is described in Section III. Expressions for these errors are developed in Sections IV to VI and are further discussed in Section VII. In Section VIII, we briefly consider the canceller performance of cascaded mismatched IF and I,Q filters.

Other types of RF-to-digital I,Q conversion errors also limit cancellation but are not considered here. Among these are I and Q sampling/strobing errors (fixed offset and random), DC bias, nonlinearities, and intermods.

II. I,Q CHANNEL ERROR MODEL

This section presents the model to be used to characterize the phase and amplitude quadrature errors and the LPF mismatches. We begin with the phase and amplitude errors. Figure 2 shows a quadrature detection model for a given channel (main or auxiliaries). Assume small (much less than one) and constant I and Q amplitude errors a_i and a_q, and phase errors ϕ_i and ϕ_q (the subscripts or superscripts i and q are used to denote the respective I and Q errors). If we denote the baseband I and Q terms before low pass filtering as $i'(t)$ and $q'(t)$, respectively, then it is straightforward to show that

$$\begin{bmatrix} i'(t) \\ q'(t) \end{bmatrix} = \left(I_2 + \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \right) \begin{bmatrix} i(t) \\ q(t) \end{bmatrix}, \tag{2.1}$$

where I_2 is the 2×2 identity matrix, $i(t)$ and $q(t)$ are the mixer inputs, and

$$\begin{aligned} \epsilon_{11} &= (1 + a_i) \cos \phi_i - 1, \\ \epsilon_{12} &= -(1 + a_i) \sin \phi_i, \\ \epsilon_{21} &= (1 + a_q) \sin \phi_q, \text{ and} \\ \epsilon_{22} &= (1 + a_q) \cos \phi_q - 1. \end{aligned} \tag{2.2}$$

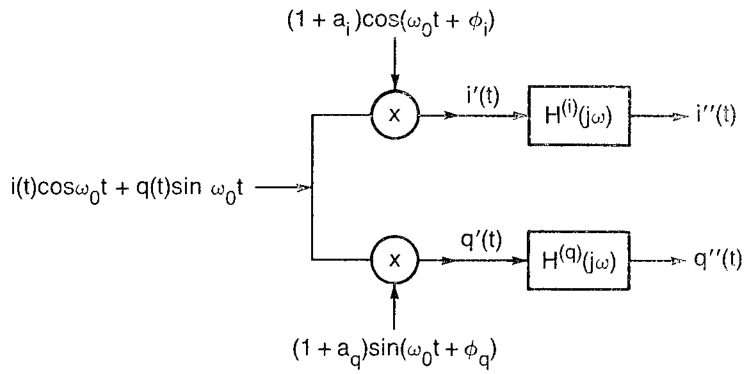

Fig. 2 — I,Q detector

Define the I,Q perturbation matrix E as

$$E = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix}. \tag{2.3}$$

Note that ϵ_{11}, ϵ_{12}, ϵ_{21}, $\epsilon_{22} << 1$. Figure 3 further depicts the I,Q amplitude and phase quadrature errors.

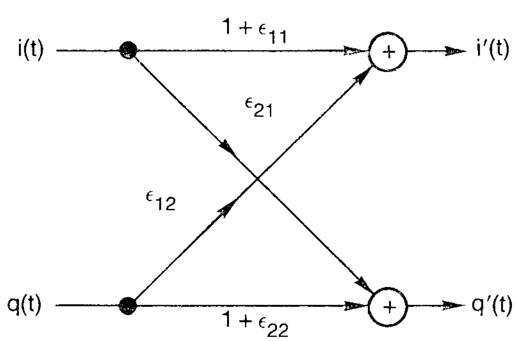

Fig. 3 — Model of I,Q quadrature errors

The LPF mismatches can be modeled by using a first order pole/zero error model that was first introduced in Ref. 1. We assume that all LPFs are designed to some desired LPF that has a frequency transfer function (FTF) denoted by $H(j\omega)$. However, because of errors in the synthesis process, the poles and zeros of $H(j\omega)$ are not as designed and have small perturbations around the desired poles and zeroes (Fig. 4). We assume that the designed filters are realizable so that all poles (unperturbed and perturbed) lie in the left-hand complex plane. This restricts the probability density function (p.d.f.) of the magnitude of the perturbation in that if λ is the maximum real part of any unperturbed pole, then the domain of the p.d.f. is bounded by $|\lambda|$, where $|\cdot|$ denotes the magnitude operation. These perturbations are assumed to be small enough that we can use first order approximations for the filter responses in the main and auxiliary channels.

It is important to note that because the LPFs are at baseband, the poles and zeros of the FTF must appear in conjugate pairs. Hence it follows that the errors of a given pole or zero must also be conjugate pairs.

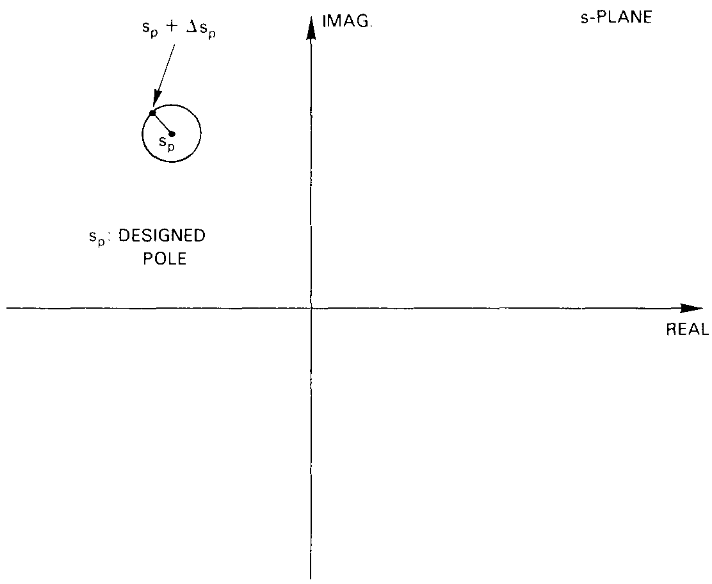

Fig. 4 — Pole with perturbation

Assume that real and imaginary parts of each perturbation are statistically independent and identically distributed zero mean real random variables. Also, assume that the perturbation each on conjugate pair is a statisically independent and identically distributed random complex variable. The variance of the magnitude of each perturbation is denoted by σ_{IQ}^2.

We assume that the desired FTF is a ratio of polynomials such that

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)}, \tag{2.4}$$

where $P(\cdot)$ and $Q(\cdot)$ are polynominals of order m and n, respectively. Consider the Laplace transform representations of $P(j\omega)$ and $Q(j\omega)$: $P(s)$ and $Q(s)$. Let $s_1^{(p)}, s_2^{(p)}, \ldots, s_m^{(p)}$ be the roots of $P(s)$, and let $s_1^{(q)}, s_2^{(q)}, \ldots, s_n^{(q)}$ be the roots of $Q(s)$. Therefore $P(j\omega)$ and $Q(j\omega)$ can be expressed as

$$P(j\omega) = (j\omega - s_1^{(p)}) \ldots (j\omega - s_m^{(p)}) \tag{2.5}$$

and

$$Q(j\omega) = (j\omega - s_1^{(q)}) \ldots (j\omega - s_n^{(q)}). \tag{2.6}$$

Consider just $P(j\omega)$. Let each root $s_k^{(p)}$, $k = 1,2,\ldots,m$ be perturbed by a small amount, $\Delta s_k^{(p)}$. Then the numerator polynomial is actually $\tilde{P}(j\omega)$, where

$$\tilde{P}(j\omega) = (j\omega - s_1^{(p)} - \Delta s_1^{(p)}) \ldots (j\omega - s_m^{(p)} - \Delta s_m^{(p)}). \tag{2.7}$$

We assume that no roots of $P(s)$ and $Q(s)$ lie on or are arbitrarily close to the $j\omega$ axis. This assumption allows us to write an expansion of $\tilde{P}(j\omega)$ and $\tilde{Q}(j\omega)$ that does not have any singular points. If we expand Eq. (2.7) and retain only the lower order terms, then

$$\begin{aligned}\tilde{P}(j\omega) &= (j\omega - s_1^{(p)})\ldots(j\omega - s_m^{(p)}) \\ &\quad - \sum_{k=1}^{m} (j\omega - s_1^{(p)})\ldots(j\omega - s_{k-1}^{(p)})\,(j\omega - s_{k+1}^{(p)})\ldots(j\omega - s_m^{(p)})\,\Delta s_k^{(p)} \qquad (2.8)\\ &= (j\omega - s_1^{(p)})\ldots(j\omega - s_m^{(p)})\left[1 - \sum_{k=1}^{m} \frac{\Delta s_k^{(p)}}{j\omega - s_k^{(p)}}\right] \\ &= P(j\omega)\left[1 - \sum_{k=1}^{m} \frac{\Delta s_k^{(p)}}{j\omega - s_k^{(p)}}\right].\end{aligned}$$

Similarly, it can be shown that the denominator polynomial when perturbed has the form

$$\tilde{Q}\,(j\omega) = Q(j\omega)\left[1 - \sum_{k=1}^{n} \frac{\Delta s_k^{(q)}}{j\omega - s_k^{(q)}}\right]. \qquad (2.9)$$

Therefore, the perturbed FTF has the form

$$\tilde{H}(j\omega) = \frac{P(j\omega)}{Q(j\omega)} \cdot \frac{1 - \displaystyle\sum_{k=1}^{m} \frac{\Delta s_k^{(p)}}{j\omega - s_k^{(p)}}}{1 - \displaystyle\sum_{k=1}^{n} \frac{\Delta s_k^{(q)}}{j\omega - s_k^{(q)}}} \qquad (2.10)$$

or

$$\tilde{H}(j\omega) = H(j\omega)\left[1 + \sum_{k=1}^{n} \frac{\Delta s_k^{(q)}}{j\omega - s_k^{(q)}} - \sum_{k=1}^{m} \frac{\Delta s_k^{(p)}}{j\omega - s_k^{(p)}}\right], \qquad (2.11)$$

where we have retained only the lower order terms.

Rewrite Eq. (2.11) as

$$\tilde{H}(j\omega) = H(j\omega)\left[1 + \sum_{k=1}^{n+m} \frac{\Delta s_k}{j\omega - s_k}\right], \qquad (2.12)$$

where we have set

$$\left.\begin{aligned} \Delta s_k &= -\Delta s_k^{(p)} \\ s_k &= s_k^{(p)} \end{aligned}\right\} \quad k = 1,2,\ldots,m \tag{2.13}$$

and

$$\left.\begin{aligned} \Delta s_{m+k} &= \Delta s_k^{(q)} \\ s_{m+k} &= s_k^{(q)} \end{aligned}\right\} \quad k = 1,2,\ldots,n\,. \tag{2.14}$$

As previously mentioned, we assume that $H_M^{(i)}$, $H_M^{(q)}$, $H_A^{(i)}$, and $H_A^{(q)}$ are designed to be matched to $H(j\omega)$, but because of inaccuracies are not equal to $H(j\omega)$. The first order pole/zero error model is used to express

$$H_M^{(i)}(j\omega) = H(j\omega)\left[1 + \sum_{m=1}^{M} \frac{\Delta s_m^{(M_i)}}{j\omega - s_m}\right], \tag{2.15}$$

$$H_M^{(q)}(j\omega) = H(j\omega)\left[1 + \sum_{m=1}^{M} \frac{\Delta s_m^{(M_q)}}{j\omega - s_m}\right], \tag{2.16}$$

$$H_A^{(i)}(j\omega) = H(j\omega)\left[1 + \sum_{m=1}^{M} \frac{\Delta s_m^{(A_i)}}{j\omega - s_m}\right], \tag{2.17}$$

and

$$H_A^{(q)}(j\omega) = H(j\omega)\left[1 + \sum_{m=1}^{M} \frac{\Delta s_m^{(A_q)}}{j\omega - s_m}\right], \tag{2.18}$$

where M is the number of poles and zeroes of $H(j\omega)$, and m is now an index. The parameters s_m, $m = 1, 2, \ldots, M$ are some ordering of the poles and zeros of $H(j\omega)$, and $\Delta s_m^{(M_i)}$, $\Delta s_m^{(M_q)}$, $\Delta s_m^{(A_i)}$, and $\Delta s_m^{(A_q)}$ are the perturbation of the poles and zeros of $H_M^{(i)}(j\omega)$, $H_M^{(q)}(j\omega)$, $H_A^{(i)}(j\omega)$, and $H_A^{(q)}(j\omega)$ respectively. These perturbations are assumed to be independent from LPF to LPF. We set

$$\Delta H_M^{(i)}(j\omega) = \sum_{m=1}^{M} \frac{\Delta s_m^{(M_i)}}{j\omega - s_m}, \tag{2.19}$$

$$\Delta H_M^{(q)}(j\omega) = \sum_{m=1}^{M} \frac{\Delta s_m^{(M_q)}}{j\omega - s_m}, \tag{2.20}$$

$$\Delta H_A^{(i)}(j\omega) = \sum_{m=1}^{M} \frac{\Delta s_m^{(A_i)}}{j\omega - s_m}, \tag{2.21}$$

and

$$\Delta H_A^{(q)}(j\omega) = \sum_{m=1}^{M} \frac{\Delta s_m^{(A_q)}}{j\omega - s_m}. \tag{2.22}$$

Thus, the first order approximations of the perturbed LPFs are

$$H_m^{(i)}(j\omega) = H(j\omega)\,(1 + \Delta H_M^{(i)}(j\omega)), \tag{2.23}$$

$$H_M^{(q)}(j\omega) = H(j\omega)\,(1 + \Delta H_m^{(q)}(j\omega)), \tag{2.24}$$

$$H_A^{(i)}(j\omega) = H(j\omega)\,(1 + \Delta H_A^{(i)}(j\omega)), \tag{2.25}$$

and

$$H_A^{(q)}(j\omega) = H(j\omega)\,(1 + \Delta H_A^{(q)}(j\omega)). \tag{2.26}$$

Finally, we depict the I,Q channel error model with canceller as shown in Fig. 5.

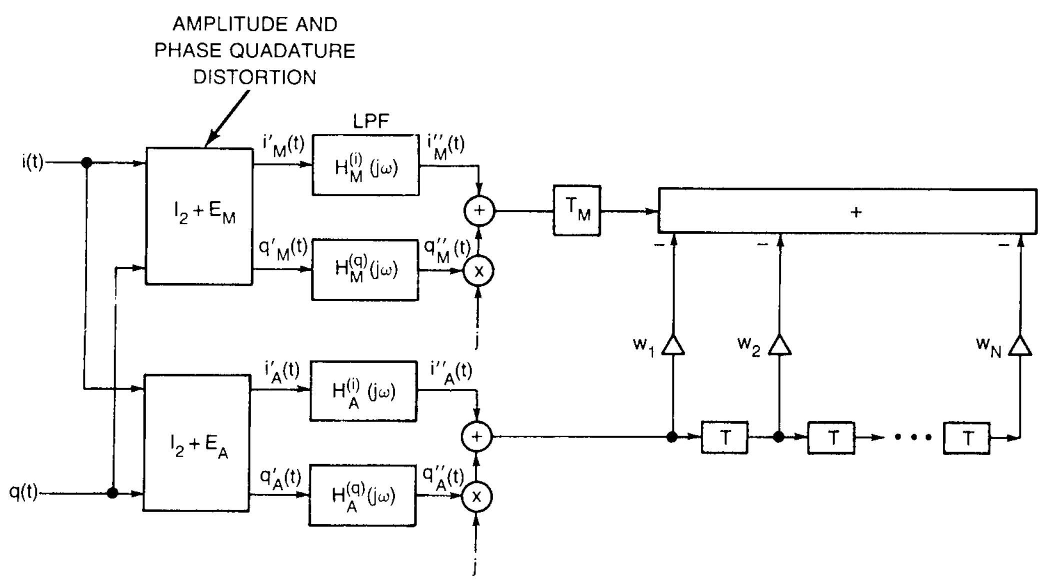

Fig. 5 — Self-cancellation model

III. SEPARATION PRINCIPLE

In the next few sections, an approximate (or first order) expression for the output noise power residue after cancellation in terms of the variance of the mismatch errors is derived. In this section, we prove a ''separation principle'' whereby we show that this error is the sum of residue terms each computed with respect to one given error; i.e., we set all errors equal to zero except one when computing a given error term.

This separation principle actually follows from a Taylor series expansion of the output noise power residue as a function of the various mismatch errors. Let $\mathbf{R}_{00}$ be the covariance matrix of auxiliary data (with time taps), let $\mathbf{r}_{00}$ be the cross correlation vector between the main channel and the auxiliary's time taps, and let $P_{\text{in}}^{(0)}$ be the input power of the main channel with $\mathbf{R}_{00}$, $\mathbf{r}_{00}$, and $P_{\text{in}}^{(0)}$ calculated with no channel errors. Let ρ_k, $k = 1, 2, \ldots, K$ represent a set of channel errors assumed to be zero mean and independent. We can write the output power residue P_{out} averaged over the channel inputs as a function of $\mathbf{R}_{00}$, $\mathbf{r}_{00}$, $P_{\text{in}}^{(0)}$, and ρ_k:

$$P_{\text{out}} = F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}, \boldsymbol{\rho}) \, , \tag{3.1}$$

where $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_K)$. In all practical cases, P_{out} is an analytical function of $\mathbf{R}_{00}$, $\mathbf{r}_{00}$, $P_{\text{in}}^{(0)}$, $\rho_1, \rho_2, \ldots, \rho_K$ with no singular values so that its functional representation can be expanded by using a multidimensional Taylor series:

$$P_{\text{out}} = F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}, \mathbf{0}) + \sum_{k=1}^{K} F_{1k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})\rho_k$$

$$+ \sum_{k=0}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}) \, |\rho_k|^2 + O(\text{cross},\Delta^3), \tag{3.2}$$

where F_{1k} $(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})$ and F_{2k} $(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})$ are the first and second order partial derivatives, respectively, of $F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})$ with respect to ρ_k and evaluated at $\boldsymbol{\rho} = \mathbf{0}$. $O(\text{cross},\Delta^3)$ represents cross terms between the ρ_k, $k = 1, 2, \ldots, K$ and/or higher order terms.

If P_{out} is averaged over the channel errors and only the first order variance terms are retained, then

$$P_{\text{out}}^{(\text{ave})} = E_{\text{errors}}\{P_{\text{out}}\} = F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}, \mathbf{0}) + \sum_{k=1}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}) \, \sigma_k^2, \tag{3.3}$$

where $\sigma_k^2 = E\{|\rho_k|^2\}$ is the variance of the kth error. Equation (3.3) exemplifies the separation principle since each $F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})$, $k = 1, 2, \ldots, K$ is computed independently of other perturbation errors.

Thus with respect to the I,Q mismatch errors, we first compute an error term for the output noise power residue that results from the LPF mismatches while setting the I,Q phase and amplitude quadrature errors equal to zero. Thereafter, we compute an error term for the I,Q phase and amplitude errors while setting the LPF errors to zero. The sum of these two errors is the first order error approximation of $P_{\text{out}}^{(\text{ave})}$.

The inverse cancellation ratio can be computed in the same manner if we assume the $F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}, \mathbf{0}) = 0$ (which is the case in the self-cancellation mode). To see this we define the inverse cancellation ratio as

$$CR^{-1} = \frac{P_{\text{out}}^{(\text{ave})}}{P_{\text{in}}^{(\text{ave})}}. \tag{3.4}$$

Note that $P_{\text{in}}^{(\text{ave})} = P_{\text{in}}^{(0,\text{ave})} + \Delta$, where $P_{\text{in}}^{(0,\text{ave})}$ is the average input power in the main channel without errors, and Δ represents a small error term caused by the channel mismatch perturbations. If $F(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)}) = 0$ and Eq. (3.3) is substituted into Eq. (3.4), then

$$CR^{-1} = \frac{\sum_{k=1}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})\sigma_k^2}{P_{\text{in}}^{(0,\text{ave})} + \Delta} \tag{3.5}$$

$$= \frac{1}{P_{\text{in}}^{(0,\text{ave})}} \sum_{k=1}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})\,\sigma_k^2 - \frac{\Delta}{(P_{\text{in}}^{(0,\text{ave})})^2} \sum_{k=1}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})\,\sigma_k^2 + O(\Delta^2).$$

Note that all terms other than the first are second order terms so that a first order approximation of CR^{-1} is

$$CR^{-1} = \sum_{k=1}^{K} F_{2k}(\mathbf{R}_{00}, \mathbf{r}_{00}, P_{\text{in}}^{(0)})\,\sigma_k^2. \tag{3.6}$$

Again the separation principle holds when computing the inverse cancellation ratio.

IV. FREQUENCY MISMATCH ERRORS

In this section, a first order expression is derived for the output noise power residue that results from having LPF frequency mismatch errors only. We use the self-cancellation configuration seen in Fig. 5. Here, $i(t)$ and $q(t)$ are assumed to be identically distributed and independent real noise sources. The noise spectrum $S_{ii}(\omega)$ and $S_{qq}(\omega)$ of the I and Q noise sources is assumed to be white so that $S_{ii}(\omega) = S_{qq}(\omega) = 1$ for all ω. For this analysis we assume that the number of delay line taps N is an odd integer.

From Fig. 5, the output residue voltage $v(t)$ can be expressed as

$$v(t) = x_M(t) - \mathbf{w}^{\mathrm{T}}\mathbf{x}(t). \tag{4.1}$$

If we set

$$P_{\text{out}} = E\{|v(t)|^2\} \tag{4.2}$$

and

$$P_{\text{in}} = E\{|x_M(t)|^2\}, \tag{4.3}$$

where P_{out} and P_{in} are the output and input noise powers, respectively, then it can be shown [2] that

$$P_{\text{out}} = P_{\text{in}} - \mathbf{w}^t \mathbf{R} \mathbf{w}, \tag{4.4}$$

where $\mathbf{R}$ is defined by Eq. (1.2) and $\mathbf{w}$ is the vector solution of Eq. (1.1). In fact, by using Eq. (1.1), we can show that

$$P_{\text{out}} = P_{\text{in}} - \mathbf{r}^t \mathbf{R}^{-1} \mathbf{r}, \tag{4.5}$$

where t denotes the complex conjugate transpose operation. The inverse cancellation ratio (or noise attenuation factor) $P_{\text{out}}/P_{\text{in}}$ can then be expressed by

$$\frac{P_{\text{out}}}{P_{\text{in}}} = \frac{P_{\text{in}} - \mathbf{r}^t \mathbf{R}^{-1} \mathbf{r}}{P_{\text{in}}}. \tag{4.6}$$

Note for the self-canceller that $P_{\text{out}}/P_{\text{in}} = 0$ if $H_A^{(i)}(j\omega) = H_M^{(i)}(j\omega)$ and $H_A^{(q)}(j\omega) = H_M^{(q)}(j\omega)$. We show this as follows. If the main and auxiliary inputs are identical, the optimal weighting $\mathbf{w}_0$ for the self-canceller is

$$\mathbf{w}_0 = (0\,0 \ldots \underset{\substack{\uparrow \\ \frac{N+1}{2} \text{ position}}}{1}\,0\,0 \ldots 0)^{\mathrm{T}}. \tag{4.7}$$

This is due to the fact that

$$x_{N_2+1}(t) = x_M(t), \tag{4.8}$$

where we have set

$$N_2 = \frac{N-1}{2}. \tag{4.9}$$

Hence, we simply subtract the N_2th output of the transversal filter seen in Fig. 5 from the output of the time delay element in the main channel to yield zero output noise power residue. As a result, if $\mathbf{r}_{00}$ and $\mathbf{R}_{00}$ are the cross covariance vector and covariance matrix under these ideal conditions (perfectly matched filters), then

$$\mathbf{R}_{00}^{-1}\mathbf{r}_{00} = \mathbf{w}_0 = (0\,0 \ldots \underset{\substack{\uparrow \\ \frac{N+1}{2} \text{ position}}}{1}\,0 \ldots 0)^{\mathrm{T}}. \tag{4.10}$$

The result of Eq. (4.10) is used to simplify many of the expressions in the upcoming derivations.

Expressions for the elements of $\mathbf{R}_{00}$ and $\mathbf{r}_{00}$ are easily derivable. It can be shown that if $R_{00,nm}$ is the nmth element of the matrix $\mathbf{R}_{00}$, then

$$R_{00,nm} = 2a\int_{-\infty}^{\infty} |H(j\omega)|^2 e^{j\omega\pi BT(n-m)} d\omega, \qquad n, m = 1, 2, \ldots, N \tag{4.11}$$

where the factor of two on the above expression results from adding the identical I and Q contributions, and a is some nonzero proportionality constant. In fact, in the following discussions we arbitrarily set $a = 1$ because we will be dealing with ratios of powers, which implies that none of the outputs calculated will be a function of a. Note that we have normalized the angular frequency to the desired angular bandwidth πB, where B is the frequency bandwidth of the desired FTF, $H(j\omega)$. Similarly, if $r_{00,n}$ is the nth element of $\mathbf{r}_0$, then

$$r_{00,n} = 2\int_{-\infty}^{\infty} |H(j\omega)|^2 e^{j\omega\pi BT(n-N_2)} d\omega, \qquad n = 1, 2, \ldots, N. \tag{4.12}$$

To be consistent with the notation of Ref. 1 from which this report derives, we define $\mathbf{R}_0$ and $\mathbf{r}_0$ (which are quantities used in Ref. 1) in terms of $\mathbf{R}_{00}$ and $\mathbf{r}_{00}$, respectively

$$\mathbf{R}_0 = \frac{1}{2}\mathbf{R}_{00}$$

and

$$\mathbf{r}_0 = \frac{1}{2}\mathbf{r}_{00}.$$

The elements of the inverse of $\mathbf{R}_0$ are defined as

$$\mathbf{R}_0^{-1} = (R_0^{(nm)}) \qquad n, m = 1, 2, \ldots, N. \tag{4.13}$$

Expressions for the elements of $\mathbf{R}$ and $\mathbf{r}$ are given by

$$\begin{aligned} R_{nm} &= \int_{-\infty}^{\infty} |H_A^{(i)}(j\omega)|^2 e^{j\omega\pi BT(n-m)} d\omega \\ &+ \int_{-\infty}^{\infty} |H_A^{(q)}(j\omega)|^2 e^{j\omega\pi BT(n-m)} d\omega, \qquad n, m = 1, 2, \ldots, N \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} r_n &= \int_{-\infty}^{\infty} H_A^{(i)*}(j\omega) H_M^{(i)}(j\omega) e^{j\omega\pi BT(n-N_2)} d\omega \\ &+ \int_{-\infty}^{\infty} H_A^{(q)*}(j\omega) H_M^{(q)}(j\omega) e^{j\omega\pi BT(n-N_2)} d\omega, \qquad n = 1, 2, \ldots, N. \end{aligned} \tag{4.15}$$

Note that the expressions for R_{nm} and r_n can be divided into the sum of individual I and Q contributions because of the independence of $i(t)$ and $q(t)$. This property is used in the forthcoming development.

If we use the first order approximations of $H_M^{(i)}(j\omega)$, $H_M^{(q)}(j\omega)$, $H_A^{(i)}(j\omega)$, and $H_A^{(q)}(j\omega)$ given by Eqs. (2.23) to (2.26) respectively, by using Eqs. (4.14) and (4.15) we can show that

$$R_{nm} = R_{00,nm} + \Delta R_{nm}^{(i)} + \Delta R_{nm}^{(q)} \qquad n,m = 1,2,\ldots,N \tag{4.16}$$

and

$$r_n = r_{00,n} + \Delta r_n^{(i)} + \Delta r_n^{(q)}, \qquad n = 1,2,\ldots,N \tag{4.17}$$

where

$$\Delta R_{nm}^{(i)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_A^{(i)} + \Delta H_A^{(i)*}) e^{j\omega\pi BT(n-m)} d\omega + \int_{-\infty}^{\infty} |H|^2 \, |\Delta H_A^{(i)}|^2 e^{j\omega\pi BT(n-m)} d\omega \tag{4.18a}$$

$$\Delta R_{nm}^{(q)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_A^{(q)} + \Delta H_A^{(q)*}) \, e^{j\omega\pi BT(n-m)} d\omega$$

$$+ \int_{-\infty}^{\infty} |H|^2 \, |\Delta H_A^{(q)}|^2 \, e^{j\omega\pi BT(n-m)} \, d\omega, \qquad n, m = 1, 2, \ldots, N \tag{4.18b}$$

and

$$\Delta r_n^{(i)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_A^{(i)*} + \Delta H_M^{(i)}) e^{j\omega\pi BT(n-N_2)} \, d\omega$$

$$+ \int_{-\infty}^{\infty} |H|^2 \, \Delta H_A^{(i)*} \, \Delta H_M^{(i)} \; e^{j\omega\pi BT(n-N_2)} \, d\omega \tag{4.19a}$$

$$\Delta r_n^{(q)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_A^{(q)*} + \Delta H_M^{(q)}) \, e^{j\omega\pi BT(n-N_2)} \, d\omega$$

$$+ \int_{-\infty}^{\infty} |H|^2 \, \Delta H_A^{(q)*} \, \Delta H_M^{(q)} \, e^{j\omega\pi BT(n-N_2)} \, d\omega, \qquad n = 1, 2, \ldots, N. \tag{4.19b}$$

Furthermore, if we define

$$P_{\text{in}} = P_{\text{in}}^{(0)} + \Delta P_{\text{in}}^{(i)} + \Delta P_{\text{in}}^{(q)}, \tag{4.20}$$

where $P_{\text{in}}^{(0)}$ is the input power when there are no perturbations, then it can be shown that

$$\Delta P_{\text{in}}^{(i)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_M^{(i)} + \Delta H_M^{(i)*}) d\omega + \int_{-\infty}^{\infty} |H|^2 \, |\Delta H_M^{(i)}|^2 d\omega \tag{4.21a}$$

and

$$\Delta P_{\text{in}}^{(q)} = \int_{-\infty}^{\infty} |H|^2 (\Delta H_M^{(q)} + \Delta H_M^{(q)*}) d\omega + \int_{-\infty}^{\infty} |H|^2 \, |\Delta H_M^{(q)}|^2 \, d\omega. \tag{4.21b}$$

We set

$$\Delta \mathbf{r} = \Delta \mathbf{r}_i + \Delta \mathbf{r}_q, \tag{4.22}$$

$$\Delta \mathbf{R} = \Delta \mathbf{R}_i + \Delta \mathbf{R}_q, \tag{4.23}$$

and

$$\Delta P_{\text{in}} = \Delta P_{\text{in}}^{(i)} + \Delta P_{\text{in}}^{(q)}, \tag{4.24}$$

where

$$\Delta \mathbf{r}_i = (\Delta r_1^{(i)}, \Delta r_2^{(i)}, \ldots, \Delta r_N^{(i)})^{\mathrm{T}}, \tag{4.25}$$

$$\Delta \mathbf{r}_q = (\Delta r_1^{(q)}, \Delta r_2^{(q)}, \ldots, \Delta r_N^{(q)})^{\mathrm{T}}, \tag{4.26}$$

$$\Delta \mathbf{R}_i = (\Delta R_{nm}^{(i)}), \qquad n, m = 1, 2, \ldots, N, \tag{4.27}$$

and

$$\Delta \mathbf{R}_q = (\Delta R_{nm}^{(q)}), \qquad n, m = 1, 2, \ldots, N. \tag{4.28}$$

We rewrite the output power residue given by Eq. (4.5) in terms of the perturbations given by Eqs. (4.22) to (4.24):

$$P_{\text{out}} = P_{\text{in}}^{(0)} + \Delta P_{\text{in}} - (\mathbf{r}_{00} + \Delta \mathbf{r})^t (\mathbf{R}_{00} + \Delta \mathbf{R})^{-1} (\mathbf{r}_{00} + \Delta \mathbf{r}). \tag{4.29}$$

Note that the $\Delta \mathbf{R}$ matrix is Hermitian Toeplitz. A second order approximation of $(\mathbf{R}_{00} + \Delta \mathbf{R})^{-1}$ is used. This can be shown to be

$$(\mathbf{R}_{00} + \Delta \mathbf{R})^{-1} = \mathbf{R}_{00}^{-1} - \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} + \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1}. \tag{4.30}$$

If Eq. (4.29) is expanded and only the second order and below perturbation terms are retained, then

$$\begin{aligned} P_{\text{out}} = {} & P_{\text{in}}^{(0)} + \Delta P_{\text{in}} - \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \mathbf{r}_{00} - \Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \mathbf{r}_{00} - \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{r} \\ & - \Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{r} + \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00} + \Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00} \\ & + \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \Delta \mathbf{r} - \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00}. \end{aligned} \tag{4.31}$$

Note that an immediate simplification of Eq. (4.31) results because

$$P_{\text{in}}^{(0)} - \mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \mathbf{r}_{00} = 0. \tag{4.32}$$

The output noise power P_{out} is averaged over the identical zero mean probability density functions (p.d.f.'s) of the pole and zero perturbations to obtain an average cancellation residue. Since the p.d.f.'s are zero mean, it follows immediately that

$$E\{\Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \mathbf{r}_{00}\} = 0 \tag{4.33}$$

and

$$E\{\mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{r}\} = 0. \tag{4.34}$$

If we substitute Eqs. (4.22), (4.23), and (4.24) into Eq. (4.31) and take the expected values over the pole/zero perturbation errors, then all of the i and q cross terms are zero or of second order because these errors are zero mean and independent. It can be shown that

$$E\{\Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{r}\} = \frac{1}{2} E\{\Delta \mathbf{r}_i^t \mathbf{R}_0^{-1} \Delta \mathbf{r}_i\} + \frac{1}{2} E\{\Delta \mathbf{r}_q^t \mathbf{R}_0^{-1} \Delta \mathbf{r}_q\} \tag{4.35}$$

$$E\{\mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00}\} = E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\} + E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} \tag{4.36}$$

$$E\{\Delta \mathbf{r}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00}\} = \frac{1}{2} E\{\Delta \mathbf{r}_i^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\} + \frac{1}{2} E\{\Delta \mathbf{r}_q^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} \tag{4.37}$$

$$E\{\mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \Delta \mathbf{r}\} = E\{(\Delta \mathbf{r}^t \mathbf{R}_0^{-1} \Delta \mathbf{R} \mathbf{R}_0^{-1} \mathbf{r}_0)^t\} \tag{4.38}$$

$$E\{\mathbf{r}_{00}^t \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \Delta \mathbf{R} \mathbf{R}_{00}^{-1} \mathbf{r}_{00}\} = \frac{1}{2} E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\}$$

$$+ \frac{1}{2} E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} + \text{second order term}. \tag{4.39}$$

By using the results from Appendix A of Ref. 1, it is straightforward that

$$E\{\Delta P_{\text{in}}^{(i)}\} = E\{\Delta P_{\text{in}}^{(q)}\} = \sigma_{IQ}^2 \Gamma_1 \tag{4.40}$$

$$E\{\Delta \mathbf{r}_i^t \mathbf{R}_0^{-1} \Delta \mathbf{r}_i\} = E\{\Delta \mathbf{r}_q^t \mathbf{R}_0^{-1} \Delta \mathbf{r}_q\} = 2\sigma_F^2 \Gamma_2 \tag{4.41}$$

$$E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\} = E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} = \sigma_{IQ}^2 \Gamma_1 \tag{4.42}$$

$$E\{\Delta \mathbf{r}_i^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\} = E\{\Delta \mathbf{r}_q^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} = \sigma_{IQ}^2 \Gamma_2 + \sigma_{IQ}^2 \Gamma_2^* \tag{4.43}$$

$$E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \Delta \mathbf{r}_i\} = E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \Delta \mathbf{r}_q\} = \sigma_{IQ}^2 \Gamma_2 + \sigma_{IQ}^2 \Gamma_3 \tag{4.44}$$

$$E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \Delta \mathbf{R}_i \mathbf{R}_0^{-1} \mathbf{r}_0\} = E\{\mathbf{r}_0^t \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \Delta \mathbf{R}_q \mathbf{R}_0^{-1} \mathbf{r}_0\} \tag{4.45}$$

$$= 2\sigma_{IQ}^2 \Gamma_2 + \sigma_{IQ}^2 \Gamma_3 + \sigma_{IQ}^2 \Gamma_3^*,$$

where σ_{IQ}^2 is the variance of the magnitude of the pole perturbation,

$$\Gamma_1 = \sum_{i=1}^{M} \int_{-\infty}^{\infty} \frac{|H|^2 d\omega}{|j\omega - s_i|^2}, \tag{4.46}$$

$$\Gamma_2 = \sum_{i=1}^{M} \sum_{k=1}^{N} \sum_{m=1}^{N} \left[R_0^{(km)} \cdot \int_{-\infty}^{\infty} \frac{|H|^2 e^{j\omega\pi BT(N_2-k)} d\omega}{j\omega - s_i} \cdot \int_{-\infty}^{\infty} \frac{|H|^2 e^{j\omega\pi BT(m-N_2)} d\omega}{-j\omega - s_i^*} \right], \tag{4.47}$$

and

$$\Gamma_3 = \sum_{i=1}^{M} \sum_{k=1}^{N} \sum_{m=1}^{N} R_0^{(km)} \cdot \int_{-\infty}^{\infty} \frac{|H|^2 e^{j\omega\pi BT(N_2 - k)}}{j\omega - s_i} d\omega \cdot \int_{-\infty}^{\infty} \frac{|H|^2 e^{j\omega\pi BT(m - N_2)}}{j\omega - s_{M-i}} d\omega. \tag{4.48}$$

Note that Γ_1 and Γ_2 are real, and the Γ_3 term results from the fact that $\Delta s_M = \Delta s_{M-m}^*$ for either the I or Q LPF.

We define $P_{\text{out}}^{(\text{ave})}$ to be P_{out} averaged over all the LPF perturbations. By substituting the result of Eqs. (4.40) to (4.45) into Eq. (4.31), it can be shown that

$$P_{\text{out}}^{(\text{ave})} = (4\Gamma_1 - 2\Gamma_2)\sigma_{IQ}^2. \tag{4.49}$$

Note that the Γ_3 term does not appear in the expression for $P_{\text{out}}^{(\text{ave})}$. Examining Eq. (4.6), we see by use of the above results that $P_{\text{out}}^{(\text{ave})}/P_{\text{in}}$ is proportional to σ_{IQ}^2 and that the constant of proportionality in the first order approximation does not change if we set $P_{\text{in}} = P_{\text{in}}^{(0)}$. We arbitrarily set

$$\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega = 1 \tag{4.50}$$

so that $P_{\text{in}}^{(0)} = 2$. Thus

$$\frac{P_{\text{out}}^{(\text{ave})}/P_{\text{in}}}{\sigma_{IQ}^2} = 2\Gamma_1 - \Gamma_2. \tag{4.51}$$

Now, $2(\Gamma_1 - \Gamma_2)$ equals the cancellation-filter mismatch ratio (CFMR) (see Eq. (3.38), Ref. 1), which is a measure of cancelling just the I (or Q) auxiliary channel against the main I (or Q) channel. Hence Eq. (4.51) is rewritten as

$$\frac{P_{\text{out}}^{(\text{ave})}/P_{\text{in}}}{\sigma_{IQ}^2} = \frac{1}{2}\ \text{CFMR} + \Gamma_1 \tag{4.52}$$

$$= \frac{1}{2}\ \text{CFMR} + \sum_{i=1}^{M} \int_{-\infty}^{\infty} \frac{|H|^2 d\omega}{|j\omega - s_i|^2}.$$

We define

$$\text{IQMF} = \sum_{i=1}^{M} \int_{-\infty}^{\infty} \frac{|H|^2\, d\omega}{|j\omega - s_i|^2} \tag{4.53}$$

to be the I,Q mismatch factor so that

$$\frac{P_{\text{out}}^{(\text{ave})}/P_{\text{in}}}{\sigma_{IQ}^2} = \frac{1}{2}\ \text{CFMR} + \text{IQMF}. \tag{4.54}$$

It turns out that CFMR can be made arbitrarily small by adjusting N and BT. However, IQMF, which is independent of N and BT, cannot. We discuss this further in Section V.

V. A SPECIAL CASE: THE BUTTERWORTH FILTER

In this section the IQMF is evaluated for the case when the desired transfer function is a Butterworth filter. This filter is of much interest because it is easily synthesized and is a low pass filter with the attenuation of the skirts controlled by the order of the filter.

This filter has the following magnitude-squared angular frequency response:

$$|H(j\omega)|^2 = \frac{c_0}{1 + \omega^{2M}}, \tag{5.1}$$

where M defines the order of the filter, the angular frequency has been normalized to the desired angular bandwidth πB, and

$$c_0 = \frac{M}{\pi} \sin \frac{\pi}{2M}. \tag{5.2}$$

The constant c_0 has been chosen so that Eq. (4.50) is satisfied. Curves of the Butterworth filter response are shown in Fig. 6 for various values of M. Note, that by increasing M that the skirts of the bandpass filter become more attenuated.

The filter is synthesized by finding an $H(s)$ function whose poles are in the left-hand side of the s-plane such that

$$H(s)H(-s)\,|_{s = j\omega} = |H(j\omega)|^2. \tag{5.3}$$

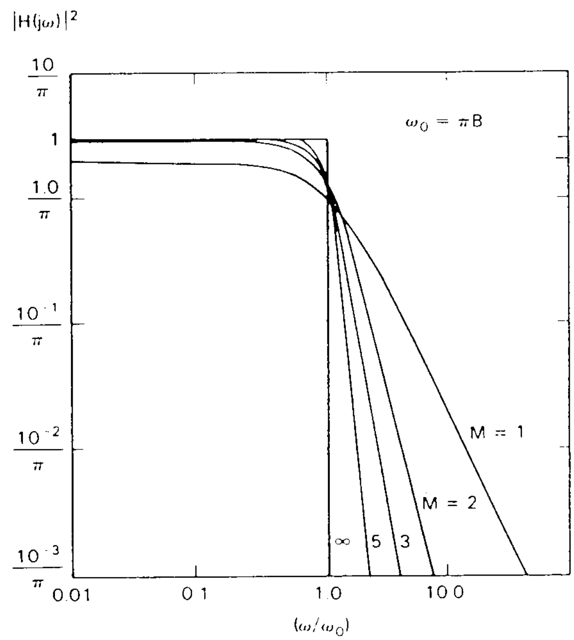

Fig. 6 — Butterworth filter response

Now the poles of $|H(j\omega)|^2$ can be shown to lie on the unit circle and are spaced equally in angle as illustrated in Fig. 7 for $M = 3$. Hence to find $H(s)$, the M left-hand plane poles of $|H(j\omega)|^2$ are identified and used to form the polynomial $H(s)$; i.e., if s_i, $i = 1, 2, \ldots, M$ are the left-handed poles, then

$$H(s) = \sqrt{c_0}[(s - s_1)(s - s_2) \ldots (s - s_M)]^{-1}. \tag{5.4}$$

As noted in the previous section, CFMR was evaluated in Ref. 1 for the Butterworth filter. In addition it is shown that

$$\int_{-\infty}^{\infty} \frac{|H|^2 d\omega}{|j\omega - s_i|^2} = \frac{\pi c_0}{2} \frac{1}{\sin \frac{\pi}{2M}(2i - 1)}. \tag{5.5}$$

Thus by using Eq. (3.53) it follows that

$$\text{IQMF} = \frac{M}{2} \sin \frac{\pi}{2M} \sum_{i=1}^{M} \csc \frac{\pi}{2M}(2i - 1), \tag{5.6}$$

where csc $(\cdot)$ is the cosecant function.

We note that IQMF is a function of M so we denote it by IQMF (M) but not by N and BT, whereas CFMR is a function of M, N, and BT which we denote by CFMR (M, N, BT). Thus

$$P_{\text{out}}^{(\text{ave})}/P_{\text{in}} = \sigma_{IQ}^2 \left[\frac{1}{2}\,\text{CFMR}(M, N, BT) + \text{IQMF}(M)\right]. \tag{5.7}$$

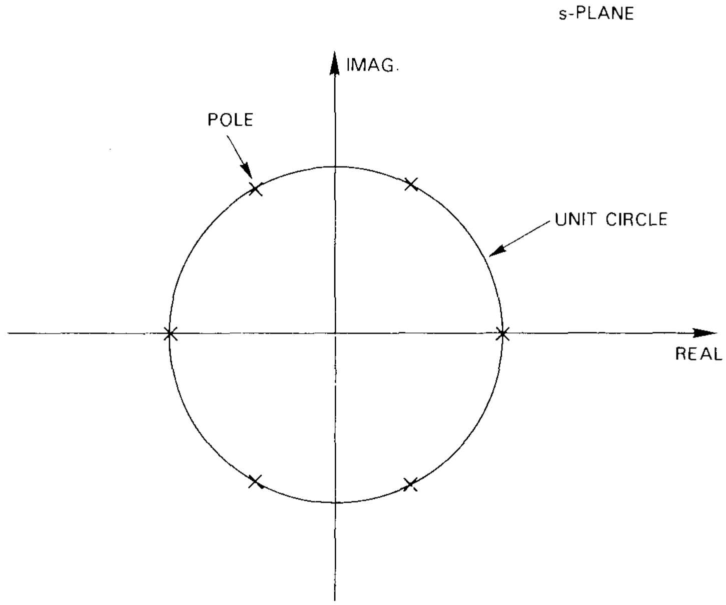

Fig. 7 — Pole plot of Butterworth filter of order 3

Plots of .5CFMR(M,N,BT) are shown in Figs. 8 to 17 for various values of M, N, and BT (N is an odd integer). In addition, for each set of plots, IQMF (M) is graphed. Note from these curves that for $N > 1$,

$$\text{IQMF}(M) >> \frac{1}{2}\,\text{CFMR}(M, N, BT). \tag{5.8}$$

Hence, the IQMF term is the dominant term of Eq. (5.7). As a result, we conclude that transversal filter compensation has a small effect on improving cancellation when I and Q frequency mismatch errors are present. This is discussed further in Section VII.

VI. PHASE AND AMPLITUDE QUADRATURE ERRORS

In this section a first order expression is derived for the inverse cancellation ratio (CR^{-1}) as a function of the phase and amplitude quadrature errors. Since it is demonstrated in the previous section that transversal filter compensation is ineffective if there are LPF mismatches, only the case when $N = 1$ is considered. As it is discussed in Section III, all the LPFs are assumed to be matched and equal to $H(j\omega)$.

For a single auxiliary canceller ($N = 1$), it can be shown that

$$\frac{P_{\text{out}}}{P_{\text{in}}} = 1 - |\alpha|^2, \tag{6.1}$$

where α is the normalized cross correlation between the main and auxiliary channels. More formally, if x_M and x_A are the main and auxiliary inputs to the canceller, then

$$|\alpha|^2 = \frac{|\overline{x_A^* x_M}|^2}{\overline{|x_A|^2}\ \overline{|x_M|^2}}, \tag{6.2}$$

where the overbar indicates that the average is taken over the random inputs. Note that $0 \leq |\alpha|^2 \leq 1$.

We assume there are statistically independent zero mean phase and amplitude quadrature errors with variances σ_ϕ^2 and σ_a^2, respectively. Furthermore, we use one of the four I and Q channels as shown in Fig. 5 as a reference channel so that this channel has no phase or amplitude errors. We choose the auxiliary Q channel to have no errors.

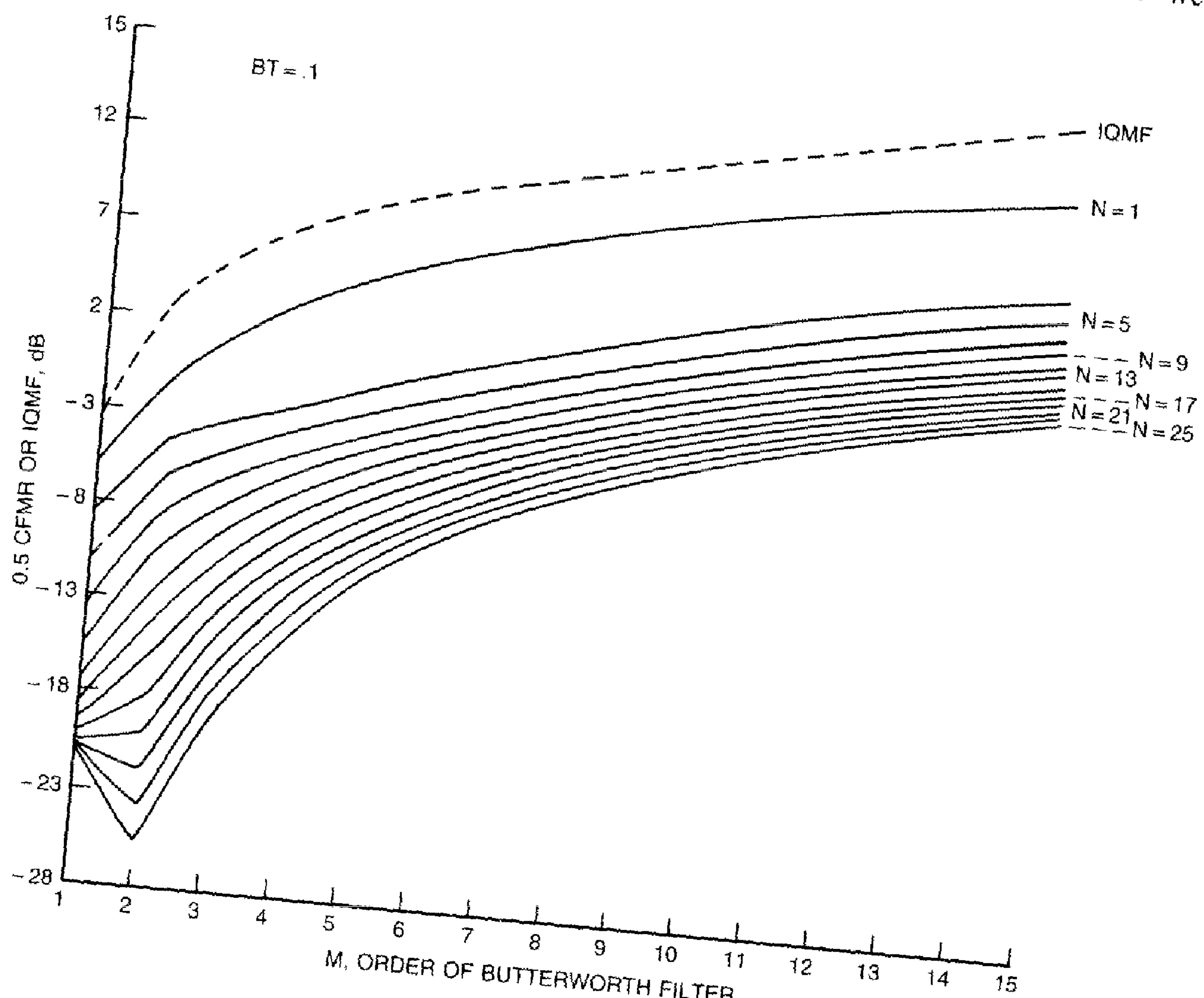

Fig. 8 — .5CFMR, IQMF vs M, BT = .1

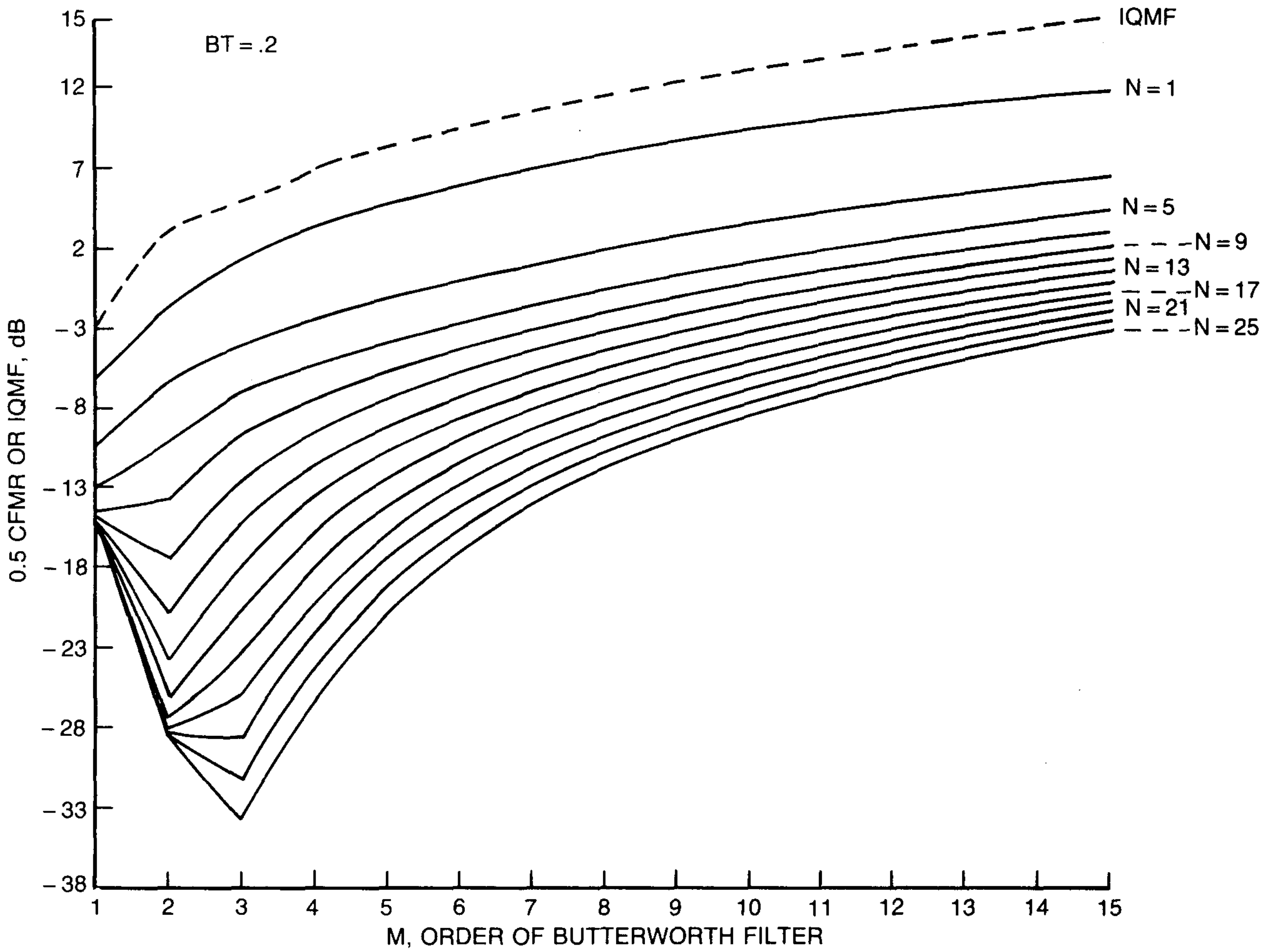

Fig. 9 — .5CFMR, IQMF vs M, BT = .2

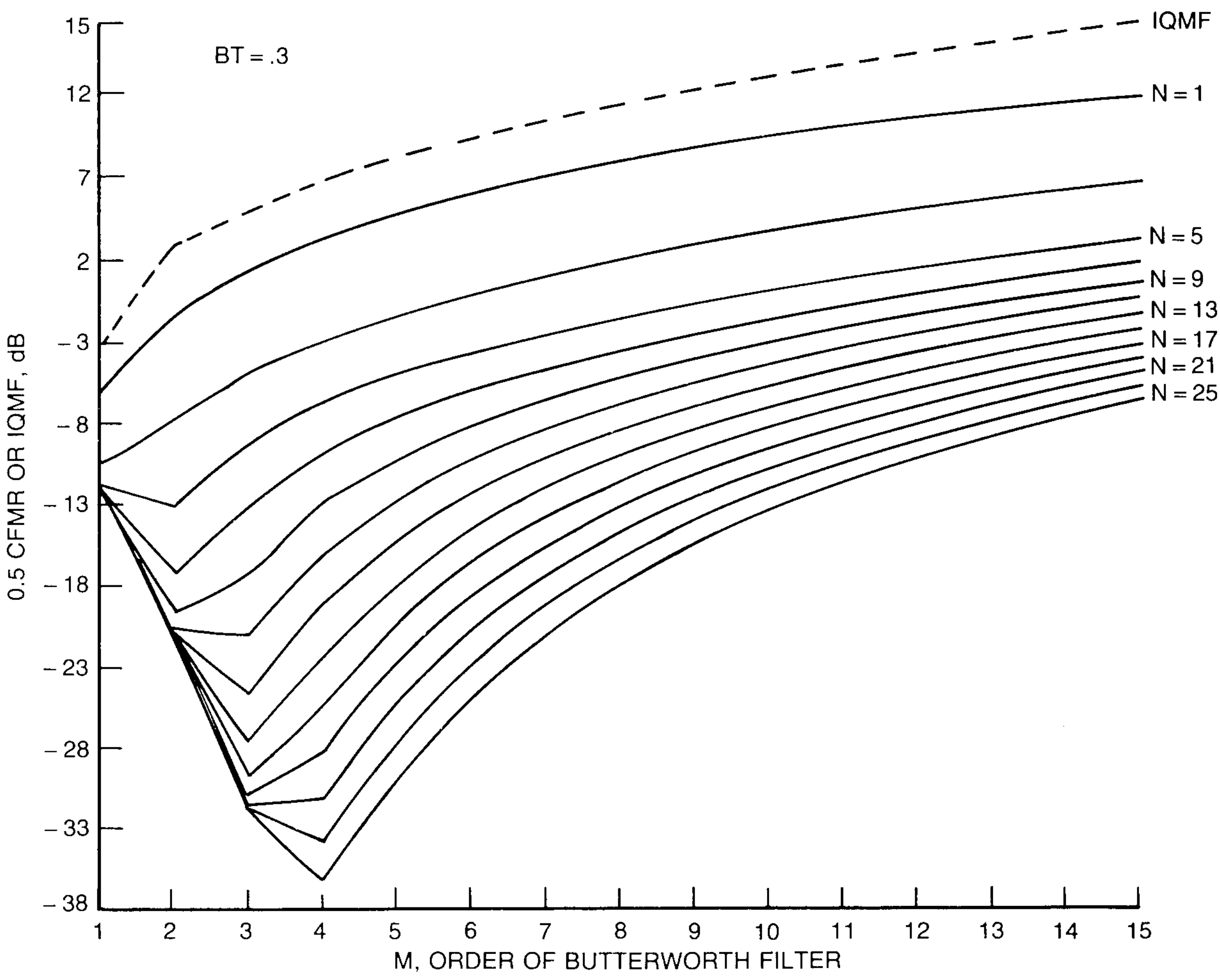

Fig. 10 — .5CFMR, IQMF vs M, BT = .3

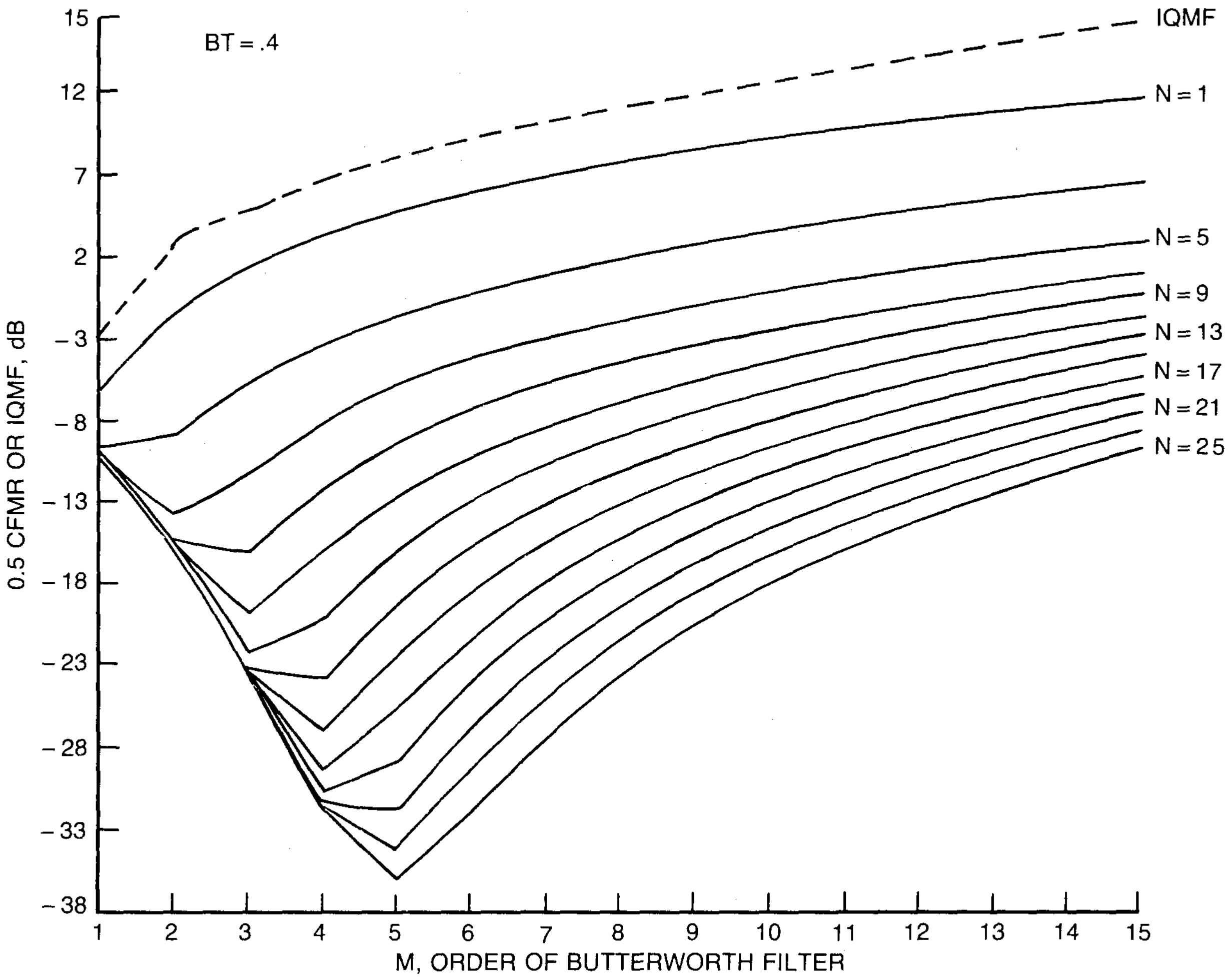

Fig. 11 — .5CFMR, IQMF vs M, BT = .4

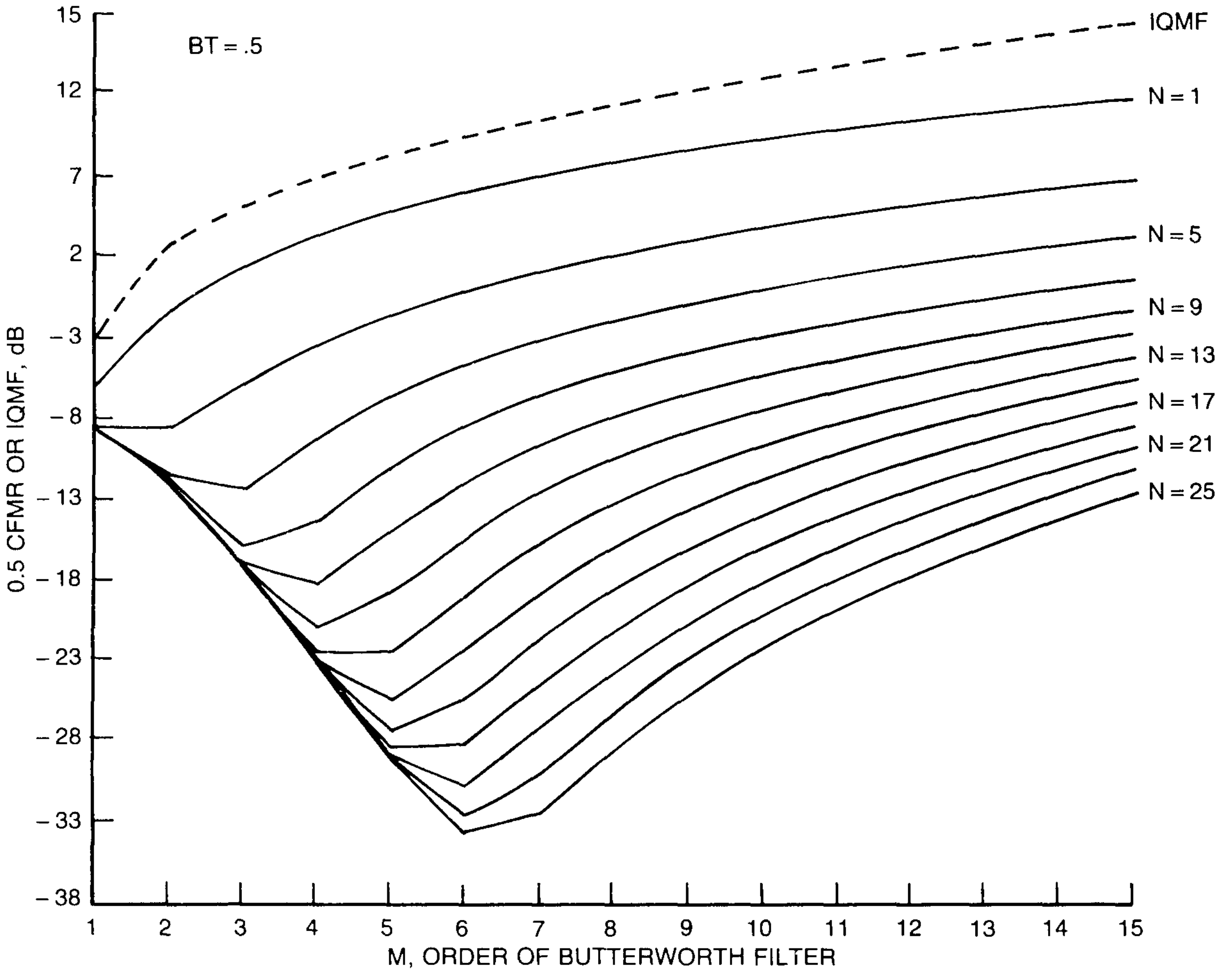

Fig. 12 — .5CFMR, IQMF vs M, BT = .5

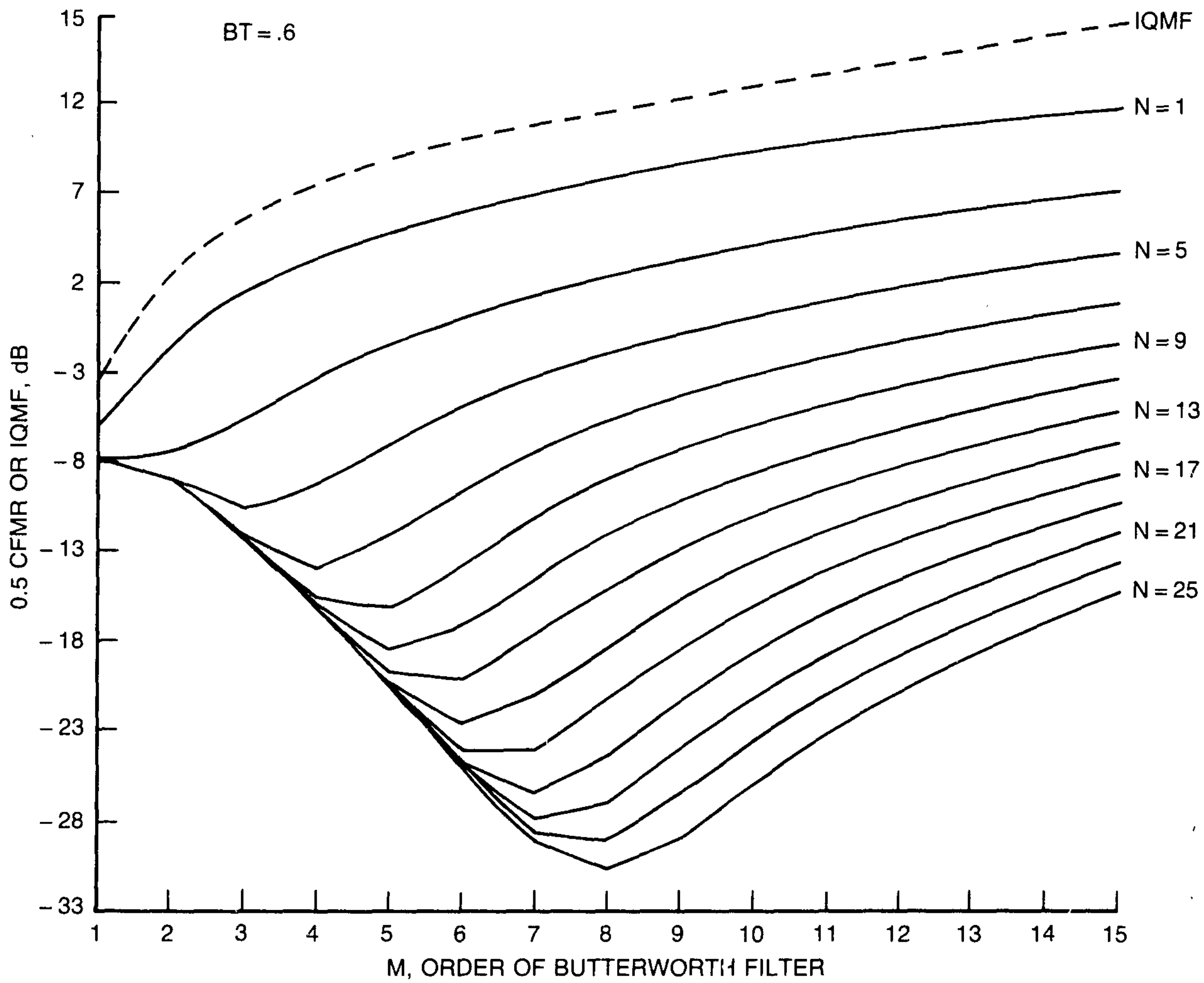

Fig. 13 — .5CFMR, IQMF vs M, BT = .6

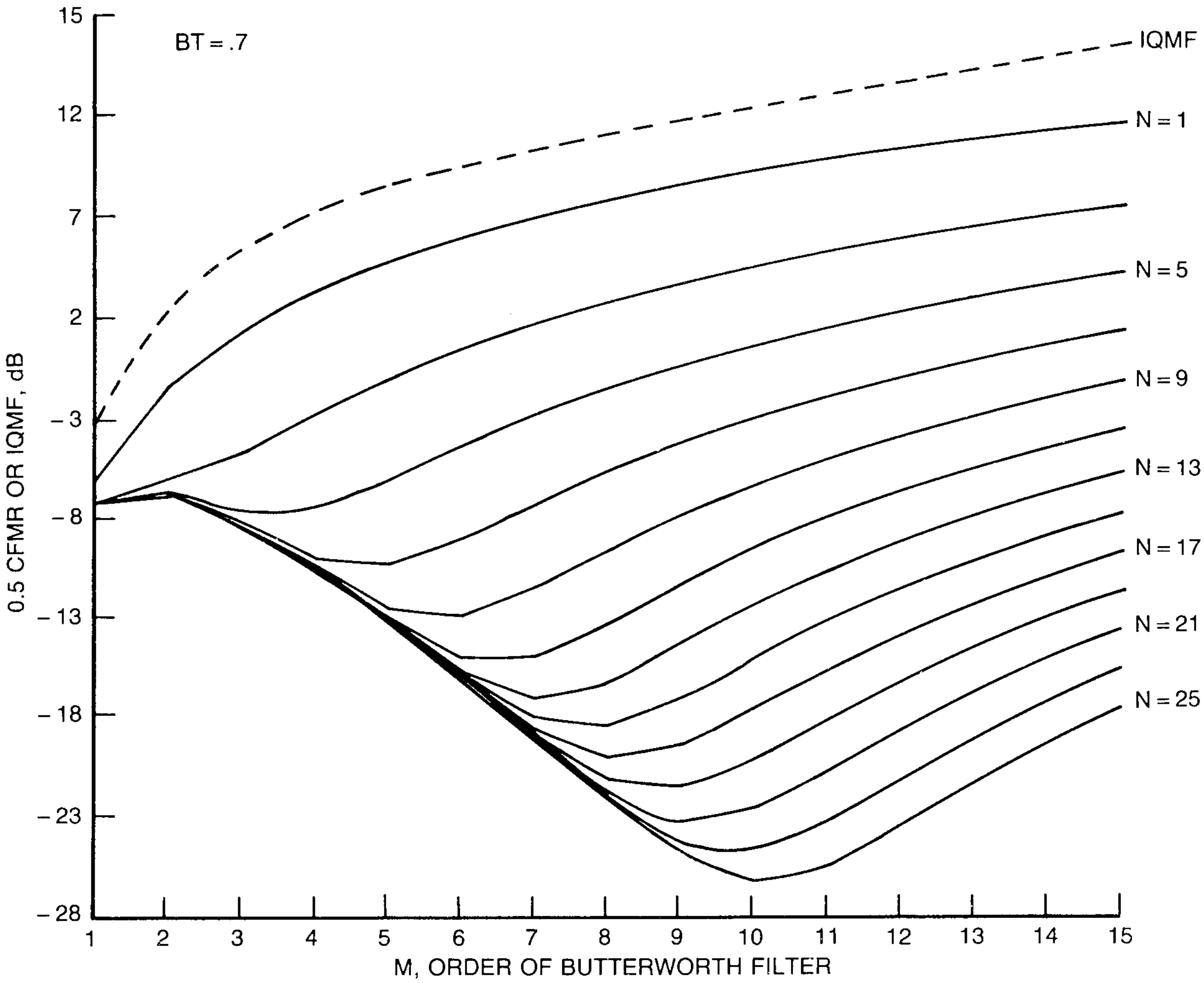

Fig. 14 — .5CFMR, IQMF vs M, BT = .7

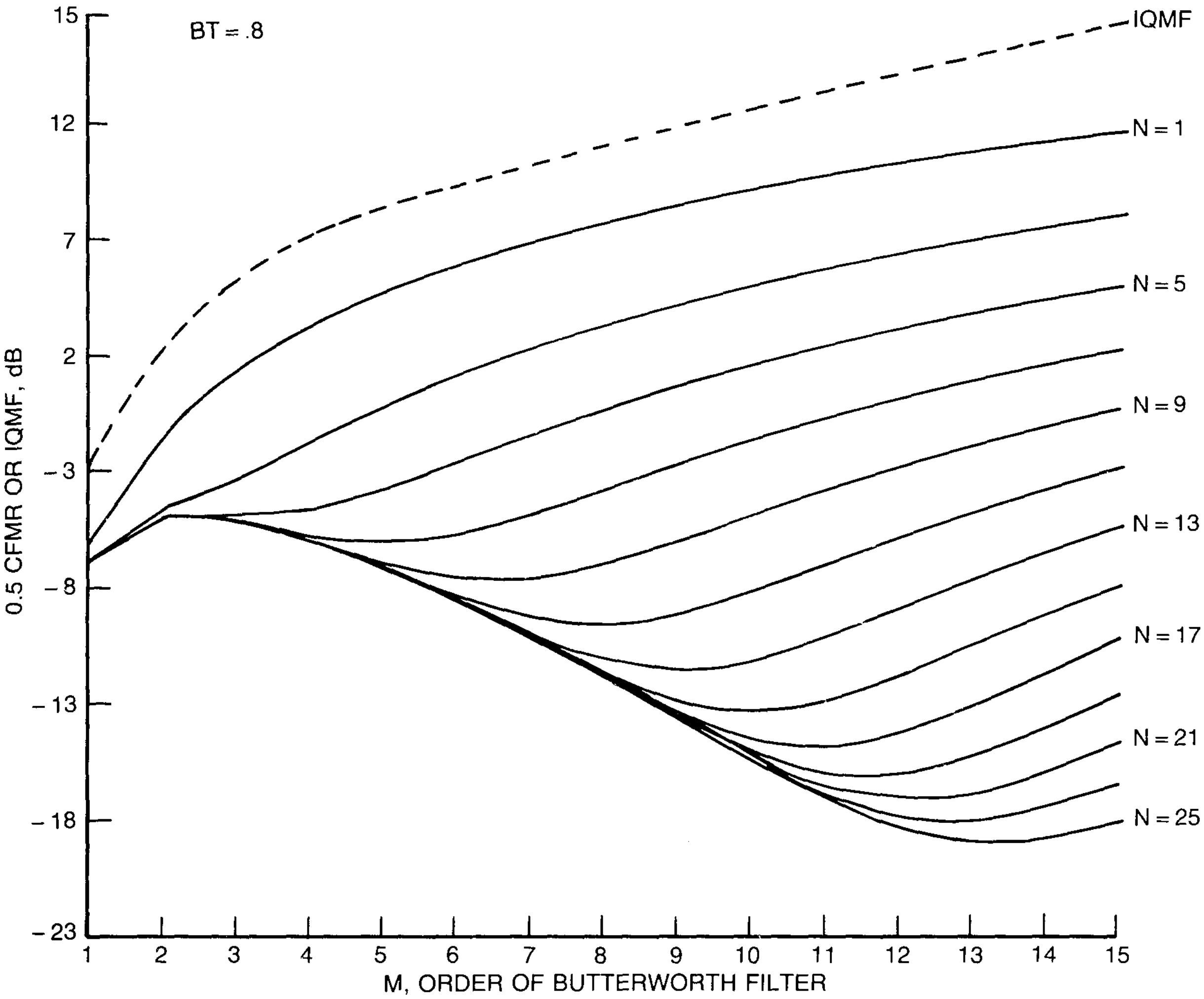

Fig. 15 — .5CFMR, IQMF vs M, BT = .8

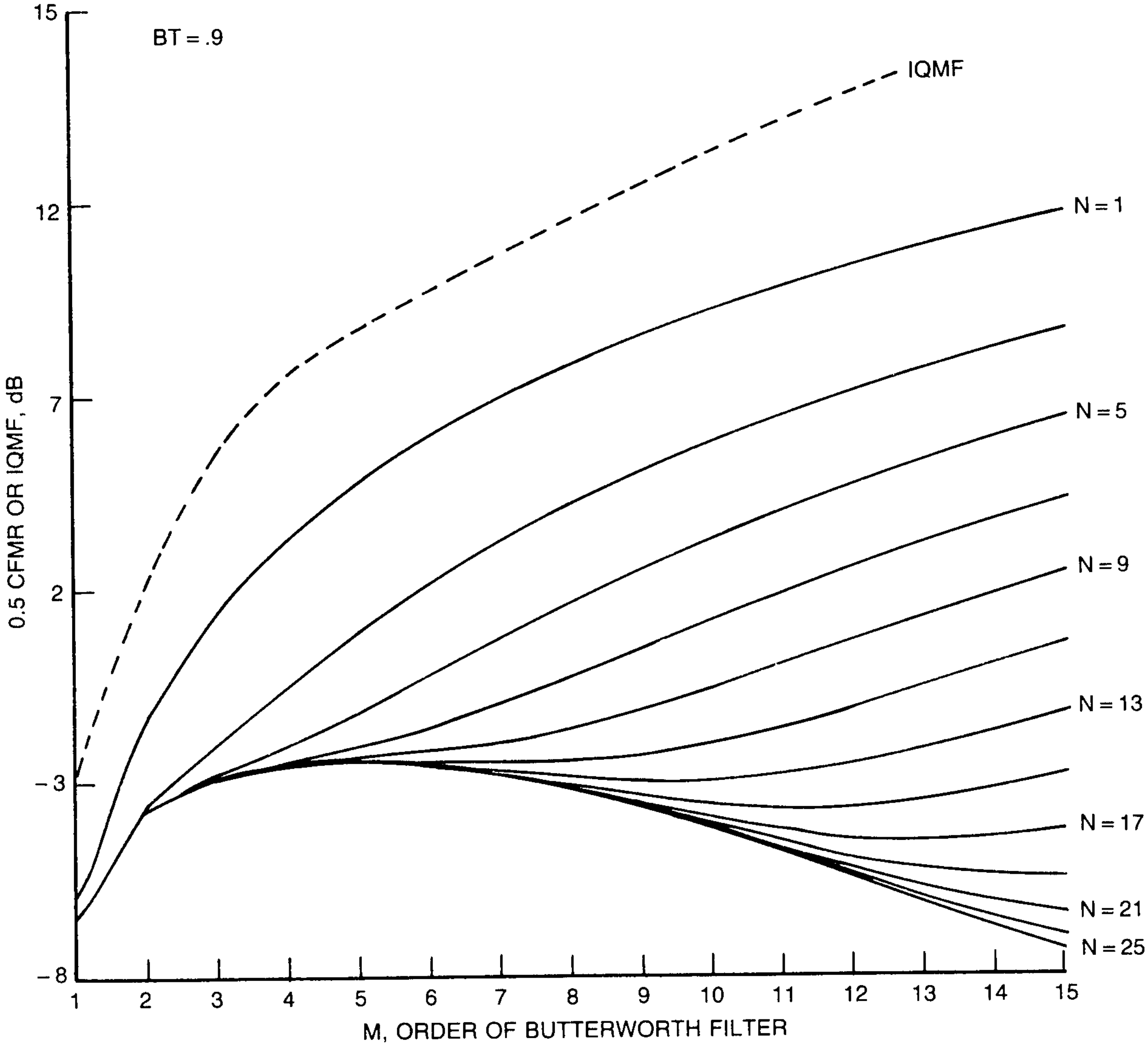

Fig. 16 — .5CFMR, IQMF vs M, BT = .9

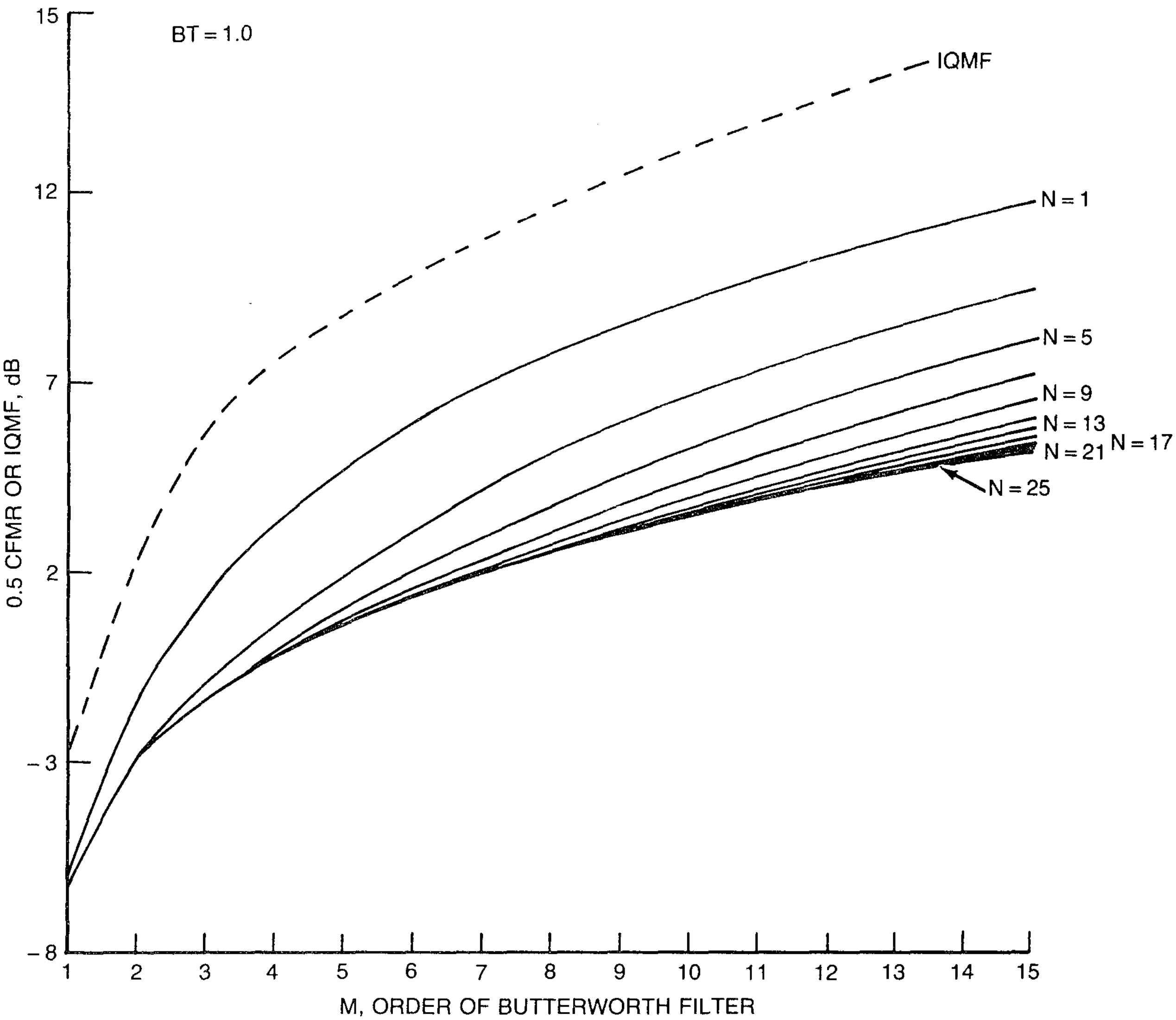

Fig. 17 — .5CFMR, IQMF vs M, $BT = 1$

Let $I(t)$ and $Q(t)$ be the output of the LPFs if there were no phase or amplitude errors in the model. Thus

$$I(t) = \int_{-\infty}^{\infty} i(\tau)h(t - \tau)\, d\tau \tag{6.3}$$

and

$$Q(t) = \int_{-\infty}^{\infty} q(\tau)h(t - \tau)d\tau, \tag{6.4}$$

where $h(t)$ is the impulse response function associated with $H(j\omega)$. Assume that $i(t)$ and $q(t)$ are identically distributed independent zero mean unit variance white noise sources.

Now assume that there are phase and amplitude quadrature errors. Thus by using the model (and variables defined) shown in Fig. 5,

$$\begin{bmatrix} i''_M(t) \\ q''_M(t) \end{bmatrix} = \left\{ I_2 + \begin{bmatrix} \epsilon_{11}^{(M)} & \epsilon_{12}^{(M)} \\ \epsilon_{21}^{(M)} & \epsilon_{22}^{(M)} \end{bmatrix} \right\} \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix} \tag{6.5}$$

$$\begin{bmatrix} i''_A(t) \\ q''_A(t) \end{bmatrix} = \left\{ I_2 + \begin{bmatrix} \epsilon_{11}^{(A)} & \epsilon_{12}^{(A)} \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} I(t) \\ Q(t) \end{bmatrix}, \tag{6.6}$$

where

$$x_M(t) = i''_M(t) + jq''_M(t) \tag{6.7}$$

and

$$x_A(t) = i''_A(t) + jq''_A(t). \tag{6.8}$$

We compute $\overline{x_A^* \, x_M}$ by using Eq. (4.50) as

$$\overline{x_A^* x_M} = (1 + \epsilon_{11}^{(M)} + j\epsilon_{21}^{(M)})\,(1 + \epsilon_{11}^{(A)}) + (\epsilon_{12}^{(M)} + j(1 + \epsilon_{22}^{(M)}))\,(\epsilon_{12}^{(A)} - j). \tag{6.9}$$

It is straightforward to show that

$$|\overline{x_A^* x_M}|^2 = 4 + \epsilon_{11}^{(M)2} + \epsilon_{11}^{(A)2} + \epsilon_{22}^{(M)2} + \epsilon_{21}^{(M)2} + \epsilon_{12}^{(M)2} + \epsilon_{12}^{(A)2} + O(\epsilon^2, 0), \tag{6.10}$$

where $O(\epsilon^2, 0)$ indicates higher order terms or terms that go to zero when the expectation is taken. In similar fashion, it can be shown that

$$\overline{|x_M|^2} = 2 + \epsilon_{11}^{(M)2} + \epsilon_{21}^{(M)2} + \epsilon_{12}^{(M)2} + \epsilon_{22}^{(M)2} + O(\epsilon^2, 0) \tag{6.11}$$

and

$$\overline{|x_A|^2} = 2 + \epsilon_{11}^{(A)2} + \epsilon_{12}^{(A)2} + O\,(\epsilon^2, 0). \tag{6.12}$$

Now

$$1 - |\alpha|^2 = \frac{\overline{|x_A|^2}\;\overline{|x_M|^2} - |\overline{x_A^* x_M}|^2}{\overline{|x_A|^2}\;\overline{|x_M|^2}} \tag{6.13}$$

and

$$\overline{|x_A|^2}\;\overline{|x_M|^2} = 4 + \Delta, \tag{6.14}$$

where $\Delta << 4$. Thus the first order expression for the expected value of $1 - |\alpha|^2$ taken over the phase and amplitude errors is given by

$$E\{1 - |\alpha|^2\} = \frac{1}{4} E\{\overline{|x_A|^2}\ \overline{|x_M|^2} - |\overline{x_A^* x_M}|^2\}. \tag{6.15}$$

By using Eqs. (6.10), (6.13), and Eq. (6.15) it can be shown that

$$E\{1 - |\alpha|^2\} = \frac{1}{4} E\{\epsilon_{11}^{(M)2} + \epsilon_{12}^{(M)2} + \epsilon_{21}^{(M)2} + \epsilon_{22}^{(M)2} + \epsilon_{11}^{(A)2} + \epsilon_{12}^{(A)2}\}. \tag{6.16}$$

Expressions for the errors in terms of the phase and amplitude quadrature errors are given by Eq. (2.2). If the small angle approximations are used that so that $\sin\phi = \phi$ and $\cos\phi = 1 - .5\phi^2$, and only first order terms are retained, then it can be shown that

$$E\{\epsilon_{11}^{(M)2}\} = E\{\epsilon_{22}^{(M)2}\} = E\{\epsilon_{11}^{(A)2}\} = \sigma_a^2 \tag{6.17}$$

$$E\{\epsilon_{21}^{(M)2}\} = E\{\epsilon_{12}^{(M)2}\} = E\{\epsilon_{12}^{(A)2}\} = \sigma_\phi^2. \tag{6.18}$$

Thus the first order expression for the inverse cancellation ratio, assuming constant phase and amplitude quadrature errors, is

$$\frac{P_{\text{out}}^{(\text{ave})}}{P_{\text{in}}} = \frac{3}{4}(\sigma_\phi^2 + \sigma_a^2). \tag{6.19}$$

VII. DISCUSSION

By use of the results from Sections V and VI we calculate the first order expression for inverse cancellation ratio with $N = 1$,

$$CR^{-1} = \frac{P_{\text{out}}^{(\text{ave})}}{P_{\text{in}}} = \frac{3}{4}\sigma_\phi^2 + \frac{3}{4}\sigma_a^2 + \sigma_{IQ}^2\left[\frac{1}{2}\,\text{CFMR}(M,1,BT) + \text{IQMF}(M)\right]. \tag{7.1}$$

To make the transversal filtering effective (for $N > 1$), it is only necessary to adaptively weight both the I and Q auxiliary channels separately as shown in Fig. 18 (this scheme is sometimes called "real weight" cancellation or "IQ weight" cancellation). Here both of the auxiliary's I and Q channels are cancelled against the main's I and Q channel.

For the IQ weighting cancellation scheme, it is easily shown that the amplitude and phase quadrature errors in the auxiliary channel can be eliminated since a $2N \times 2N$ nonsingular matrix exists that transforms the $2N$ auxiliary taps (I and Q) into $2N$ outputs that have no phase and amplitude errors. By use of the IQ weighting scheme shown in Fig. 18, it is straightforward to show that the first order inverse cancellation ratio for IQ weighting is given by

$$CR_{IQWT}^{-1} = \sigma_{IQ}^2\ \text{CFMR}(M,N,BT). \tag{7.2}$$

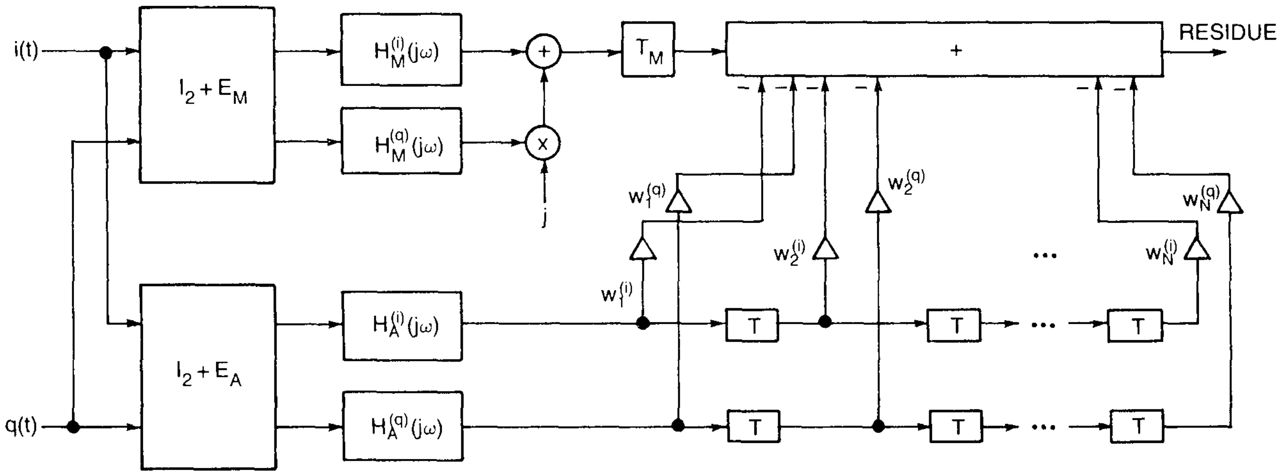

Fig. 18 — IQ weighting canceller

Plots of .5CFMR are given in Figs. 8 to 17. Reference 1 discusses in detail methodologies for choosing optimal values of M, N, and BT under given constraints. Note that although using a single adaptive transversal filter does not effectively compensate for the I,Q mismatches, it does compensate for IF filter mismatches even though the I,Q detecter may be part of the receiver channels. In this case, only the IF filter mismatches are equalized, not the I,Q LPF mismatches.

VIII. CASCADED IF/IQ FILTER CONSIDERATIONS

A receiver chain may have an IF section as shown in Fig. 1(a) for tuning purposes and thus inherent in its functioning is IF frequency filtering. In the previous section, we showed that I,Q adaptive weighting is necessary to make transversal filtering effective. We now consider the inverse CR that results from using I,Q weighting for when the input channels pass through mismatched IF filters followed by (or cascaded with) one of the I,Q LPFs.

Figure 19 shows the channel model. We assume that the LPFs are designed to an ideal FTF, $H_{IQ}(j\omega)$, and that the IF filters are designed to $H_{IF}(j\omega)$. The rms errors for the pole positions of the IF and IQ filters are given by σ_{IF} and σ_{IQ} respectively. Let the bandwidth of the IF filter be B_{IF}, and let that of the IQ filter be B_{IQ}. We define a parameter β to be the ratio of B_{IF} to B_{IQ}, or

$$\beta = \frac{B_{IF}}{B_{IQ}}. \tag{8.1}$$

Note that β is always greater than one. Let M_{IF} and M_{IQ} be the number of poles and zeros in the IF and IQ filters, respectively. Let $s_i^{(IF)}$, $i = 1, 2, \ldots, M_{IF}$ and $s_i^{(IQ)}$, $i = 1, 2, \ldots, M_{IQ}$ be the poles and zeroes of these filters. Assume that all poles have been normalized by πB_{IQ}. For example, if all the IQ poles are equal, then they all lie on the unit circle in the complex s-plane.

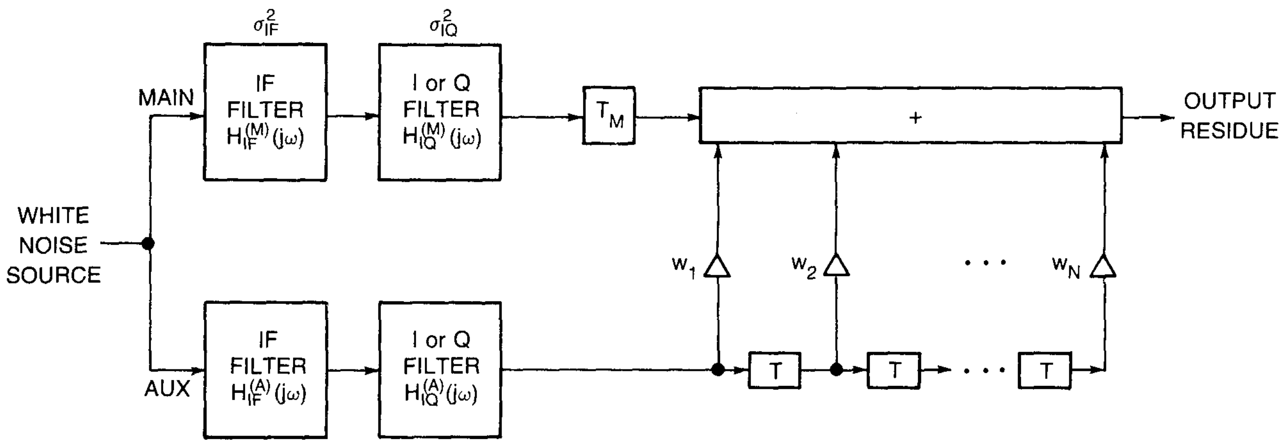

Fig. 19 — Cascaded IF/IQ filter and canceller modeling

By using the separation principle described in Section III, we can show for the cascaded IF/IQ filters that

$$CR^{-1} = \sigma_{IF}^2 \left[\sum_{i=1}^{M_{IF}} \int_{-\infty}^{\infty} \frac{|H_{IF}|^2 \, |H_{IQ}|^2 \, d\omega}{|j\omega - s_i^{(IF)}|^2} \right. \tag{8.2}$$

$$- \sum_{k=1}^{N} \sum_{m=1}^{N} R_0^{(km)} \int_{-\infty}^{\infty} \frac{|H_{IF}|^2| \; |H_{IQ}|^2 \, e^{j\omega\pi B_{IQ} T(N_2-k)} d\omega}{j\omega - s_i^{(IF)}}$$

$$\left. \cdot \int_{-\infty}^{\infty} \frac{|H_{IF}|^2 \, |H_{IQ}|^2 \, e^{j\omega\pi B_{IQ} T(m-N_2)} d\omega}{-j\omega - s_i^{(IF)*}} \right]$$

$$+ \sigma_{IQ}^2 \left[\sum_{i=1}^{M_{IQ}} \int_{-\infty}^{\infty} \frac{|H_{IF}|^2 \, |H_{IQ}|^2 \, d\omega}{|j\omega - s_i^{(IQ)}|^2} \right.$$

$$- \sum_{k=1}^{N} \sum_{m=1}^{N} R_0^{(km)} \int_{-\infty}^{\infty} \frac{|H_{IF}|^2 \, |H_{IQ}|^2 \, e^{j\omega\pi B_{IQ} T(N_2-k)} d\omega}{j\omega - s_i^{(IQ)}}$$

$$\left. \cdot \int_{-\infty}^{\infty} \frac{|H_{IF}|^2 \, |H_{IQ}|^2 \, e^{j\omega\pi B_{IQ} T(m-N_2)} d\omega}{-j\omega - s_i^{(IQ)*}} \right],$$

where $R_0^{(km)}$ is the k,m element of the inverse of the matrix whose k,m element $R_{0,km}$ is defined by

$$R_{0,km} = \int_{-\infty}^{\infty} |H_{IF}|^2 \; |H_{IQ}|^2 \cdot e^{j\omega\pi B_{IQ}(k-m)} d\omega. \tag{8.3}$$

We do not give any results that can be derived from this complicated expression other than to point out that if β goes to infinity, it is straightforward to show that the multiplicative factor of σ_{IF}^2 seen in Eq. (8.2) goes to zero. Hence if the IF filters are much more difficult to match over B_{IQ} or equivalently $\sigma_{IF}^2 >> \sigma_{IQ}^2$, then significantly widening the bandwidth of the IF filter can ameliorate the effects of IF filter mismatch. For very wide IF bandwidths, if $\beta >> 1$, then the cascaded inverse CR^{-1} is determined only by the mismatches in the I,Q LPFs. Of course widening this bandwidth is not without consequences. For example, by doing so the input noise power into the I,Q detectors is increased so that the dynamic range of these detectors must be increased.

IX. SUMMARY

The effects of I and Q phase, amplitude, and LPF errors on adaptive cancellers were investigated. I,Q errors occur because of errors in the synthesis process of the mixers and LPFs designed to be identical for each input channel. These I,Q errors among the channels result in cancellation degradation. Using a separation principle developed and discussed in the text, first order expressions for cancellation degradation as a function of phase, amplitude quadrature errors, and LPF mismatches were obtained. The separation principle was proved whereby we showed that a first order expression for cancellation degradation can be computed that is the sum of the individual cancellation degradations each computed with respect to one given error. In effect, we set all errors equal to zero except one when computing a given degradation.

Tapped delay line transversal filters have been proposed as a way to compensate for LPF frequency mismatch and thus improve cancellation performance. However, it was shown that if there is any LPF frequency mismatch, then transversal filtering has a small effect on improving canceller performance. The method of individual I,Q adaptive transversal filter weighting was suggested as a means of eliminating the phase and amplitude errors and of making the canceller performance responsive to transversal filter compensation. In addition, the cancellation performance of cascaded mismatched IF and I,Q filters was briefly considered.

X. REFERENCES

1. Karl Gerlach, "Adaptive Canceller Limitations Due to Frequency Mismatch Errors," NRL Report 8949, Jan. 1986.

2. R.A. Monzingo and T.W. Miller, *Introduction to Adaptive Arrays* (John Wiley and Sons, 1980).

3. W.F. Gabriel, "Adaptive Digital Processing Investigation of DFT Subbanding vs Transversal Filter Canceller," NRL Report 8981, July 1986.